From

Erdös

To

Kiev

Problems of Olympiad Caliber

Ross Honsberger

The Mathematical Association of America

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The Association, for its part, was delighted to accept the gracious gesture initiating the revolving fund for this series from one who has served the Association with distinction, both as a member of the Committee of Publications and as a member of the Board of Governors. It was with genuine pleasure that the Board chose to name the series in her honor.

The books in the series are selected for their lucid expository style and stimulating mathematical content. Typically, they contain an ample supply of exercises, many with accompanying solutions. They are intended to be sufficiently elementary for the undergraduate and even the mathematically inclined high-school student to understand and enjoy, but also to be interesting and sometimes challenging to the more advanced mathematician.

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This book is dedicated to the memory of
two tireless benefactors of mathematics
— **Léo Sauvé and Fred Maskell** —
the founding editors of the problems journal
Crux Mathematicorum

Contents

Preface

There is no doubt that a little friendly competition has brought a great deal of enjoyment into the world. At advanced levels of competition, however, there is a temptation to sacrifice aesthetic values to power in an all-out drive for results. With the regional, national and international mathematics competitions heating up over the years, many of our most capable young students today are so intent on gaining command of a multitude of techniques that they are in danger of developing habits of mind which can lead to lasting disability in the appreciation of the beauty of elementary mathematics. I have often entertained the thought that the major peril in studying for a PhD is that reading for the degree is so intensely acquisitive over a long period that there is a substantial risk that one's ability to read will survive only in the emaciated form of being able to "skim for ideas". I would like to think that coaches never tire of reminding our young mathematics competitors that the real reason great mathematicians can work so hard and accomplish so much is their overwhelming fascination with the subject. It can't be easy for the leaders of training programs to guard against fostering the view that the worth of an idea lies in its utility. One doesn't hear nearly often enough these days of anyone having spent a delightful evening curled up with an exciting math book. Even a good elementary problem can be captivating and its ingenious solution positively thrilling. These are the feelings I hope the present collection will engender in the reader. The problems are generally pretty difficult, and while an olympiad candidate might learn something from them, my only real interest is in sharing the beauties of elementary mathematics with the general reader.

I came across the great majority of these problems in the Olympiad Corner columns of the 1987 and 1988 volumes of the problems journal *Crux*

Mathematicorum, which is published by the Canadian Mathematical Society. It is a periodical that is unsurpassed of its kind and a great credit to all who contribute to its excellence. Although many of my solutions may proceed along common lines, considerably more than half the solutions in the present collection are of my own invention. When a solution is the work of others, credit is invariably given in the course of the exposition; however, since I have written things up to suit myself, they are not responsible for the shortcomings in presentation.

I would like to extend my deepest thanks to Don Albers for his unflagging encouragement and support over many years. It is also a pleasure to thank the former chairman Joe Buhler and the members of the Dolciani Editorial Board for their perceptive reviews of the manuscript. Finally, warm thanks are due Elaine Pedreira and Beverly Ruedi for their technical expertise and careful handling of the manuscript through the publication process.

Seven Solutions by George Evagelopoulos

At the present time (1995) George Evagelopoulos practices criminal law in Athens, Greece. He loves mathematics and in his student days, when these solutions were fashioned, he spent a good deal of his spare time doing problems. Here is a sample of his many solutions that have appeared in *Crux Mathematicorum* over the years. The references given are to *Crux Mathematicorum*.

George is still active mathematically and is currently the Editor-in-Chief of the Greek edition of the outstanding journal *Quantum*.

1. Problem 1

(This first problem comes from the 1983 Australian Olympiad. It was also solved by C. Cooper, Central Missouri State University, and by John Morvay, Dallas, Texas [1985, 71])

In a large urn (Grecian, of course) there are 75 white balls and 150 black ones, and beside the urn is a big pile of black balls. Now, the following two-step operation is performed repeatedly. First, two balls are withdrawn at random from the urn and then

(a) if they are both black, one of them is put back and the other is thrown away,
(b) if one is black and the other white, the white one is put back and the black one is thrown away,
(c) if they are both white, they are both thrown away and a black ball from the pile is put into the urn.

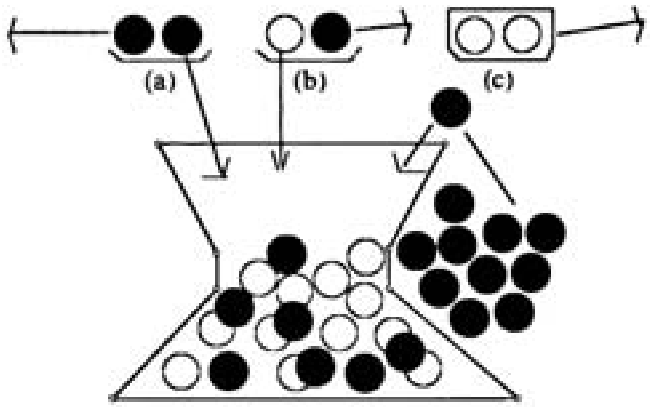


FIGURE 1

Therefore, whatever the case, at each stage two balls are removed from the urn and only one is put back, thus reducing the number of balls in the urn by one. Eventually, then, the urn will reach the point of containing just a single ball. The question is "What color is this last ball?"

Solution

It is easy to see that the number of black balls in the urn always changes by one at each stage (a loss in cases (a) and (b), and a gain in case (c)). Since the total number of balls is inexorably decreasing to one, the number of black balls must eventually fall through all the values 150, 149, . . . , 1, and perhaps even zero, though not likely without some fluctuations. Since we don't know whether there might be some white balls around when the number of black balls reaches one, this discovery is not enough to resolve the problem. A look at the way in which the number of white balls can change, however, settles things immediately; it's too bad we didn't think of considering this in the first place.

The number of white balls is unchanged by cases (a) and (b), and it drops by two under (c). Hence the *parity* of the number of white balls is always the same. Since it was odd to begin with, namely 75, it must remain odd throughout, ensuring at least one white ball in the urn at all times. Therefore the last ball can't avoid being white.

On the other hand, if the number of white balls at the beginning had been even, it would be impossible to wind up with a single white ball (an odd number), implying the last ball would be black.

2. The Cube and the Checkerboard

(From the Moscow Olympiad, 1973 [1990, 35])

This problem concerns a 50×50 checkerboard and a cube whose faces are the same size as the squares on the board. Beginning in the lower left corner, the cube is successively rolled about one of its bottom edges to travel from square to square around the board, winding up finally in the opposite corner. At each move, the cube is allowed to roll only to the *right* or *up* the board. Even with these restrictions, there are still a great many ways to shuffle the 49 steps to the right with the 49 upward steps necessary to reach the opposite corner (in fact, there are $\binom{98}{49}$ ways, a number of 29 digits).

Now suppose this cube is actually a die, each face holding one of the numbers 1, 2, 3, 4, 5, 6, arranged as usual so that the numbers on opposite faces add up to 7. Moreover, suppose that, whenever the cube lands on a square, the number on its bottom face is printed on that square. Since the rules do not allow the cube to return to a square that is already numbered, when all the rolling is done, a total of $1 + 49 + 49 = 99$ squares will carry an integer. What is the greatest and least values of the sum S of these 99 integers?

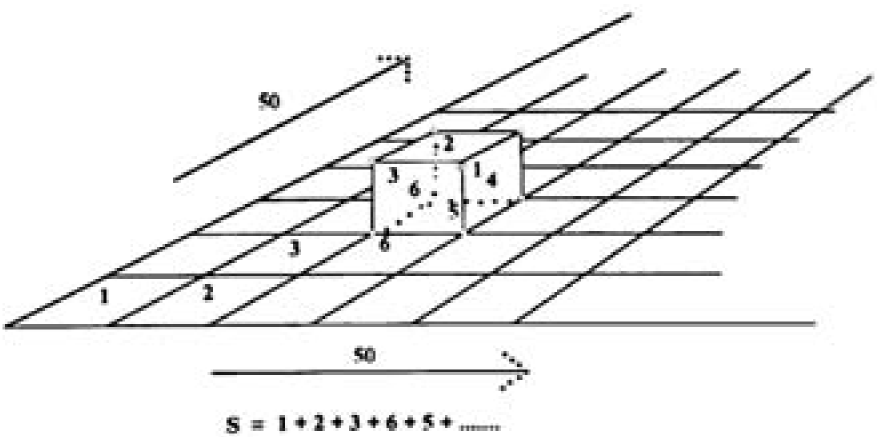


FIGURE 2

Solution

Trying to imagine and keep track of even the first few rolls in a typical path requires uncommon powers of visualization. However, it is not difficult to

picture the process sufficiently clearly to discover the key to the problem in the following simple observation:

> since you can't turn back to the left or move down the board, before a bottom face can print itself *again* on another square of the board, it must first *rise to the top* of the cube.

Now, when a face containing the integer x is on the top of the cube, the opposite face is printing the complementary integer $7-x$ at the bottom of the cube. Because a face must come to the top before printing again, somewhere between every two occurrences of the integer x, there must occur a $7-x$; and conversely, somewhere between every two occurrences of $7-x$, the number x must occur. That is to say, in every path $P = \{a_1, a_2, \ldots, a_{99}\}$, the complementary pair $(x, 7-x)$ alternate along the sequence at various intervals, beginning with either number of the pair as the case may be. At the end of all these pairs $(x, 7-x)$, there might or might not occur an extra x or $7-x$ as the first number in an *incomplete* final pair.

$$\ldots\overbrace{x\ldots 7-x}\ldots\overbrace{x\ldots 7-x}\ldots---\ldots\overbrace{x\ldots 7-x}\ldots$$

$$\ldots\overbrace{7-x\ldots x}\ldots\overbrace{7-x\ldots x}\ldots---\ldots\overbrace{7-x\ldots x}\ldots 7-x\ldots$$

Thus, for example, to the right of all the complementary pairs $(1, 6)$, there can occur in the sequence either no further occurrence of 1 or 6, or precisely the first member of the pair, be it a 1 or a 6. Similarly for the other complementary pairs $(2, 5)$ and $(3, 4)$. Since there cannot be more than three incomplete pairs, at most three of the 99 numbers can fail to be bound up in entire pairs.

Now, obviously these pairs occupy an *even* number of the 99 places in the sequence, implying that there must be an *odd* number of incomplete pairs. Consequently, there must be either one or three incomplete pairs, resulting respectively from 49 or 48 entire pairs.

Since the numbers in each pair add up to 7, 49 pairs would contribute to the sum S a total of $49 \cdot 7 = 343$, which the single unpaired extra integer would increase by either 1, 2, 3, 4, 5, or 6. In this case, then, S ranges from 344 to 349.

However, 48 pairs would contribute $48 \cdot 7 = 336$ to S, and the three remaining integers, coming one from each of the pairs $(1, 6)$, $(2, 5)$, $(3, 4)$, could not increase S by more than $6 + 5 + 4$ nor by less than $1 + 2 + 3$, for a maximum of $336 + 15 = 351$ and a minimum of $336 + 6 = 342$.

It is an easy exercise to confirm that these limits are actually attainable (this is left to the reader), giving the conclusion that the required extrema are in fact 351 and 342.

Isn't it remarkable that the enormous number of values of S,

$$\binom{98}{49} = 25477612258980856902730428600,$$

all fall in a narrow band of only 10 integers?

3. Problem M1056

(*Kvant*, 1987, due to A. S. Merkuriev [1990, 104])

Suppose that each entry in a 1987×1987 matrix M is a real number of magnitude not exceeding 1. Suppose also that these entries have been carefully chosen and arranged so that the four entries in every 2×2 submatrix add up to zero. Prove, then, that the sum of all the entries in M cannot exceed 1987.

Solution

Since every 2×2 submatrix can be discarded in adding up the entries in M, one's first impulse might be to throw away as many 2×2 submatrices as possible and hope that not more than 1987 entries are left; if successful, the desired conclusion would then be at hand since every entry has magnitude ≤ 1. Beginning in the bottom left corner, then, one might start cutting out rows and columns of 2×2 sections until the entire 1986×1986 submatrix in that corner is discarded. Unfortunately, this still leaves intact the whole first row and last column of M, containing $2 \cdot 1987 - 1$ entries, which is far more than desired, suggesting that this approach might not be as easy to carry out as we had hoped.

Surely, though, the elimination of a great many zero-sum 2×2 submatrices must turn out to be fundamental in any solution to the problem. The difficulty would appear to lie in finding a decomposition of M which, after discarding all the 2×2 submatrices, leaves a remnant whose entries, however many there might be, can easily be shown to add up to not more than 1987. George's solution is a beautiful gem!

Considering the very first entry of M separately, the rest of M can be partitioned into $\frac{1}{2}(1987 - 1) = 993$ L-shaped pieces, each of width 2 and of lengths $3, 5, 7, \ldots, 1987$, respectively, as illustrated. The arms of these L-shaped pieces can be further divided into abutting 2×2 sections which extend along each arm to a corner piece A which is just a 3×3 section with the upper left

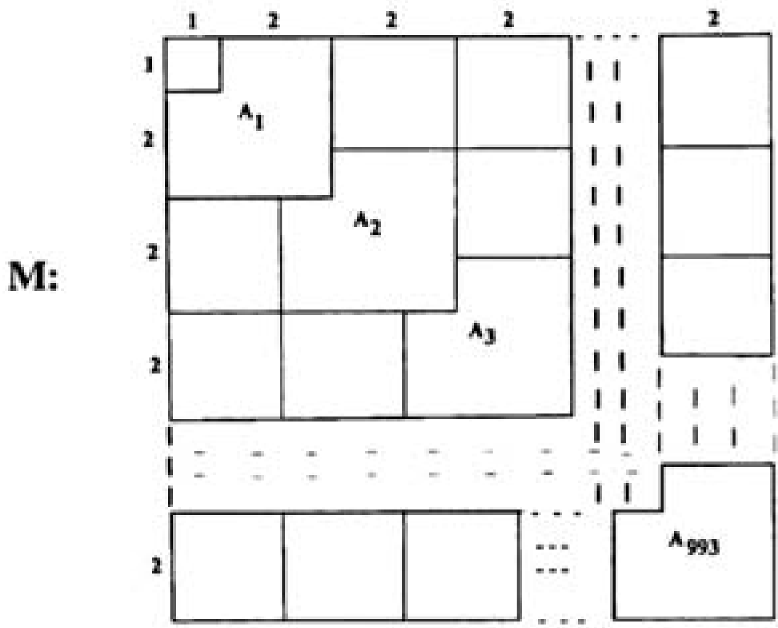


FIGURE 3

corner missing. Thus the sum S of all the entries in M is the value of its first entry plus the sums in the 993 corner-pieces A_i.

But it is not difficult to see that the sum of the entries in any corner-piece A_i cannot exceed 2:

A_i:

	a	b
e	d	c
f	g	h

$$\begin{aligned}
&\text{the sum of the entries in } A_i\\
&= (a+b+c+d+) + e + f + g + h\\
&= \quad 0 \quad + (e+f+g+d) + h - d\\
&= \quad 0 \quad + \quad 0 \quad + h - d\\
&= h - d\\
&\le 2, \quad \text{since} \quad |h|,\ |d| \le 1.
\end{aligned}$$

Thus $S \le 1 + 993(2) = 1987$, as desired.

4. Kürschák Competition

(Hungary, 1983 [1989, 230])

Consider the polynomial $f(x)$ whose first and last coefficients are 1 and whose intervening coefficients a_i are all nonnegative:

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + 1.$$

If the equation $f(x) = 0$ happens to have n real roots, is it not remarkable that the value of $f(2)$ must then be at least 3^n?
Prove this unlikely consequence: $f(2) \geq 3^n$.

Solution

Since all the coefficients a_i are ≥ 0, the substitution of any nonnegative number for x would make $f(x)$ at least 1, implying that all the roots of $f(x) = 0$ must be negative numbers, say $-r_1, -r_2, \ldots, -r_n$. Using these roots to factor $f(x)$, we obtain

$$\begin{aligned} f(x) &= x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + 1 \\ &= (x + r_1)(x + r_2)(x + r_3)\cdots(x + r_n) \\ &= x^n + (r_1 + r_2 + \cdots + r_n)x^{n-1} + (r_1r_2 + r_1r_3 + \cdots)x^{n-2} + \cdots + r_1r_2\cdots r_n. \end{aligned}$$

Thus each coefficient a_k is given by

$$\begin{aligned} a_k &= \text{the sum of the } \tbinom{n}{k} \text{ products of the roots taken } k \text{ at a time} \\ &= \sum r_{i_1}r_{i_2}\cdots r_{i_k}. \end{aligned}$$

Also, the absolute term $r_1r_2\cdots r_n = 1$.

Now, applying the arithmetic mean–geometric mean inequality to the $\binom{n}{k}$ terms that make up a_k, we obtain

$$\frac{a_k}{\binom{n}{k}} = \frac{\sum r_{i_1}r_{i_2}\cdots r_{i_k}}{\binom{n}{k}} \geq \left[\prod r_{i_1}r_{i_2}\cdots r_{i_k}\right]^{1/\binom{n}{k}},$$

from which we get

$$a_k \geq \binom{n}{k}\left[\prod r_{i_1}r_{i_2}\cdots r_{i_k}\right]^{1/\binom{n}{k}}.$$

Admittedly, this doesn't seem to be very promising. However, this awkward product melts away completely with the brilliant observation that, since there is no preference for one r_i over another, it follows that each r_i occurs in the product the same number of times altogether, with the result that

$$\begin{aligned} \prod r_{i_1}r_{i_2}\cdots r_{i_k} &= (r_1r_2\cdots r_n)^t, \quad \text{for some positive integer } t, \\ &= 1^t = 1. \end{aligned}$$

Therefore,

$$a_k \geq \binom{n}{k}.$$

Now, k only runs from 1 to $n-1$, and so if we set $a_0 = a_n = 1$, then a_0 and a_n would respectively equal $\binom{n}{0}$ and $\binom{n}{n}$, making

$$a_k \geq \binom{n}{k} \quad \text{for all} \quad k = 0, 1, 2, \ldots, n.$$

Thus we can write

$$f(x) = \sum_{k=0}^{n} a_k x^{n-k},$$

and the value of $f(2)$ is given by $\sum_{k=0}^{n} a_k 2^{n-k}$. Since $a_k \geq \binom{n}{k}$, then

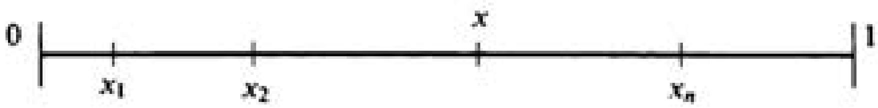
$$\begin{aligned} f(2) &\geq \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 2^{n-k} \\ &= (1+2)^n \quad \text{by the binomial theorem} \\ &= 3^n, \quad \text{as desired.} \end{aligned}$$

5. Problem 5

(Now for another problem from Australia [1985,70] also solved by V. N. Murty, Pennsylvania State University).

No matter what n real numbers $x_1, x_2, \ldots, x_n$ may be selected in the closed unit interval $[0,1]$, prove that there always exists a real number x in this interval such that the average unsigned distance to the x_i is exactly $\frac{1}{2}$:

$$\frac{1}{n} \sum_{i=1}^{n} |x - x_i| = \tfrac{1}{2}.$$



Solution

Adopting a perfectly straightforward approach, George looks at the function

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} |x - x_1|.$$

At $x = 0$, we have

$$f(0) = \frac{1}{n}\sum_{i=1}^{n} |-x_i| = \frac{1}{n}\sum_{i=1}^{n} x_i \quad (\text{since } x_i \geq 0).$$

For $x = 1$, we get

$$\begin{aligned} f(1) &= \frac{1}{n}\sum_{i=1}^{n} |1 - x_i| \\ &= \frac{1}{n}\sum_{i=1}^{n} (1 - x_i) \quad (\text{since } x_i \in [0, 1]) \\ &= \frac{1}{n}\left(n - \sum_{i=1}^{n} x_i\right) \\ &= 1 - f(0), \end{aligned}$$

and we have the crucial relation

$$f(0) + f(1) = 1.$$

In view of this, the two values $f(0)$ and $f(1)$ are either each equal to $\frac{1}{2}$, providing two solutions to the problem, or their values *straddle* $\frac{1}{2}$, in which case the continuity of the function implies $f(x) = \frac{1}{2}$ for some x between 0 and 1.

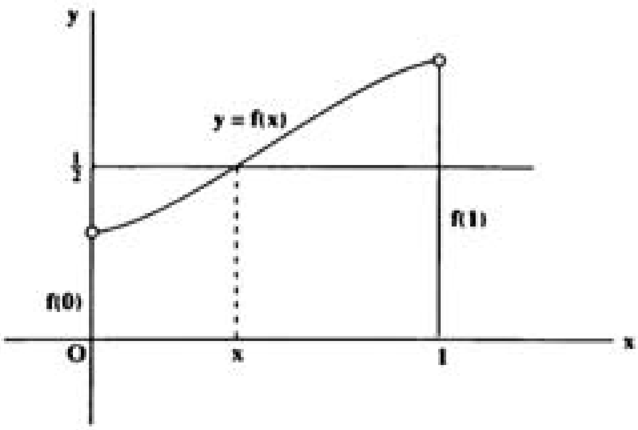


FIGURE 5

6. Problem M1043 from *Kvant*

(*Kvant*, 1987, due to S. V. Konyagin [1990, 102])

In this problem we are asked to decide whether it is possible to partition the entire set of integers $(\ldots, -1, 0, 1, 2, \ldots)$ into three disjoint subsets in such a way that, for every n, the integers n, $(n-50)$, and $(n+1987)$ are always one in each of the three subsets.

Solution

Let's not prejudge the issue, but try to construct the desired partition. It will be convenient to summarize our progress with the following notation:

> let $m \sim k$ signify that m and k must belong to the same subset, and let (m, k, t) denote that m, k, and t belong one to each of the three different subsets; thus (m, k, t) implies, for example, that t does not occur in the same subset with either m or k.

Let's begin with the easy deductions of two basic results which must hold for any successful way of partitioning the integers:

(a) $n \sim (n + 1937)$ (i.e., for every integer n, the integers n and $(n + 1937)$ must always occur together in the same subset) and
(b) $n \sim (n - 150)$.

(a) $n \sim (n + 1937)$:
Since the fundamental property $(n - 50, n, n + 1987)$ must hold for all integers n, we have, by replacing n respectively by $n - 50$ and $n + 1987$, that

$$(n - 100, n - 50, n + 1937) \quad \text{and} \quad (n + 1937, n + 1987, n + 2 \cdot 1987).$$

The former shows that $(n + 1937)$ cannot be in the same subset with $(n - 50)$, and the latter shows that it isn't in with $(n + 1987)$ either. But, in view of the basic $(n - 50, n, n + 1987)$, it must be in with one of the three numbers $(n - 50)$, n, or $(n + 1987)$. By elimination, then, it must be that $(n + 1937)$ is with n, itself.

That is to say, for every integer n, the entire arithmetic progression $n, (n + 1937), (n + 2 \cdot 1937), \ldots$ must lie in the same subset. In particular, the entire progression $(0, 1937, 2 \cdot 1937, \ldots)$ must be all together in the same subset.

We shall make use of this general result of part (a) in establishing the second property, $n \sim (n - 150)$, which similarly claims that the progression $n, (n - 150), (n - 2 \cdot 150), \ldots$, always occurs entirely in the same subset (in particular, the progression $[(646 \cdot 150 - 50), (645 \cdot 150 - 50), (644 \cdot 150 - 50), \ldots]$).

(b) $n \sim (n - 150)$:

We have already noted that $(n - 100, n - 50, n + 1937)$ must hold. Since n is always in with $(n + 1937)$, this gives $(n - 100, n - 50, n)$. Holding for all n, this in turn yields $(n - 150, n - 100, n - 50)$, showing that $(n - 150)$ is never in with either $(n - 100)$ or $(n - 50)$. In view of the established $(n - 100, n - 50, n)$, it follows that $(n - 150)$ must always be with n.

Now then, you must be wondering what's so special about these numbers 1937 and 150. The puzzling answer is that

$$50(1937) = 646(150) - 50 \qquad (= 96850).$$

Because of the exclusiveness of the particular arithmetic progressions noted above, we have the vital relations

$$\begin{aligned} 0 &\sim 1937 \sim 2\cdot 1937 \sim \cdots \sim 50\cdot 1937 = 646(150) - 50 \sim 645(150) - 50 \\ &\sim 644(150) - 50 \sim \cdots \sim 0(150) - 50 = -50, \end{aligned}$$

showing that 0 and -50 must be together in the same subset. But, for $n = 0$, this is in clear violation of the mandatory $(n - 50, n, n + 1987)$, implying the proposed partition is impossible!

There may be various ways of solving this problem, but doesn't George's solution display the most remarkable ingenuity? We conclude this little collection with the following easy problem.

7. Problem M1057 from *Kvant*

(*Kvant*, 1987 [1990,105])

Suppose two players, first A and then B, take turns writing down positive integers subject to the two rules

(i) no integer many exceed an agreed upon limit L, and
(ii) no integer may be a *divisor* of a number already used.

The first one who is unable to play is the loser.

For example, for a limit of $L = 10$, a game might proceed as follows:

A starts with 10 (eliminating all of 1, 2, 5, 10 from play,
leaving 3, 4, 6, 7, 8, and 9 still available);

B plays 4 (not eliminating anything but the 4 itself,
leaving 3, 6, 7, 8, and 9 available);

A replies with 7 (leaving 3, 6, 8, and 9);
and B plays 8 (leaving just 3, 6, and 9).

Now, if A were silly enough to play 6, eliminating the 3, B would win with 9; but A can win with the 3, for then both 6 and 9 can be played, either one by B and the other by A.

Therefore it is possible for either player to win. Prove, however, that for every limit L, there exists a winning strategy for A.

Solution

We emphasize that we are not required to *find* a winning strategy for A, but simply to show that one must exist. Since somebody must win (they can't both be the *first* who is unable to play), *if* we can show that there is no winning line of play for B, then there must be some line of play that A can adopt which B is unable to handle successfully, that is, there is some winning strategy for A. We proceed indirectly.

Suppose, to the contrary, then, that there is a winning strategy for B. In this case B would be able to parry any first move made by A. Now, the number 1, being a divisor of every integer, can only be played on A's very first move if it is ever to be played at all. B's strategy must be able to handle this, say by replying with n, leaving A to continue from the set $\{2, 3, \ldots, n-1, n+1, \ldots, L\}$, reduced by any additional divisors that n might have.

On the other hand A is free to begin with any integer up to L, and he might choose to start with this particular integer n. Unfortunately, this first move would saddle B with the same set of losing options that B's own strategy foists upon A in the case when A begins with 1. Thus a winning strategy for B necessarily contains the seeds of its own downfall, and is therefore an untenable notion.

It is interesting to note that an awareness of the existence of a winning strategy is rather cold comfort for A. Although it guarantees ultimate success in the quest for a winning strategy, it provides no hint how to find one.

A Decomposition of a Triangle

1. An Easy Warm-up

If you were asked to subdivide a given square into 4 squares, you wouldn't be long thinking of the obvious checkerboard quartering. But if 2, 3, or 5 sub-squares had been requested, you would have been bound to fail because each of these cases is impossible. Prove, however, that these are the only exceptions, that is,

> prove that a given square can be decomposed into n squares, not necessarily all the same size, for all $n = 4, 6, 7, 8, \ldots$.

A simple quartering of any sub-square obviously increases the number of squares by three (four are gained but the original square is lost); thus, if a decomposition containing n squares is attainable, so is the whole infinite family

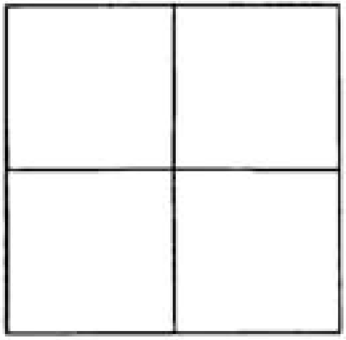


FIGURE 6

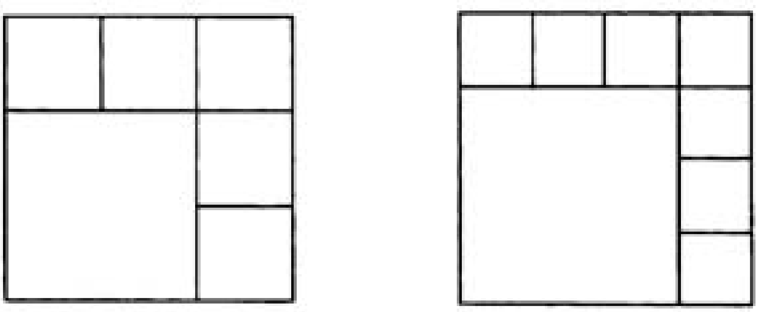
of cases $\{n, n+3, n+6, n+9, \ldots\}$. Consequently, if one can figure out how to achieve the three cases of $n = 4, 6$, and 8, all the cases in question will be covered:

$$\{4, 7, 10, 13, \ldots\}, \{6, 9, 12, \ldots\}, \{8, 11, 14, \ldots\}.$$

We have already seen how to do $n = 4$, and so it remains only to consider 6 and 8.

But these also turn out to be minor problems. If each of two adjacent sides of the given square is divided into k equal parts and a strip of k equal squares is built against each of these sides, a border of $2k - 1$ little squares is obtained (this row and column of squares overlap in a corner square). With the complementary square that remains of the given figure, a decomposition containing $2k$ sub-squares is produced. Thus our goal is reached simply by taking $k = 3$ and 4.

k = 3 k = 4



n = 6 n = 8

FIGURE 7

2. The Triangle Problem

The corresponding problem of decomposing a triangle into n sub-triangles is solved trivially by any pencil of $n - 1$ lines across the triangle from a vertex, and so some additional condition must be demanded in order to give substance to the problem. Indeed, this time we are required to have only *isosceles* triangles in the decomposition. The decomposition may not be possible if only two or three triangles are called for, but it strikes me as remarkable that every triangle has an isosceles-triangle decomposition for all $n \geq 4$.

> Prove that a given triangle can always be decomposed into n isosceles triangles for every positive integer $n \geq 4$.

This appeared as part of Problem 200 in *Crux Mathematicorum* [1976, 220] and the following clever solution by Gali Salvatore of Ottawa (alias Léo Sauvé) was published in 1977 (134–135). You might also enjoy the closely related Problem 1115, posed in 1986, p. 27 and solved in 1987, p. 189:

Determine the positive integers n for which there exists a decomposition of an equilateral triangle into n equilateral triangles.

While two of the altitudes might lie outside a triangle, the altitude to the longest side always lies strictly inside the triangle. If AD is an interior altitude of given triangle ABC and E and F are the midpoints of the other two sides, then AD, DE, and DF subdivide ABC into four isosceles triangles (the midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices since it is the circumcenter of the triangle). Thus any triangle can be decomposed into four isosceles triangles.

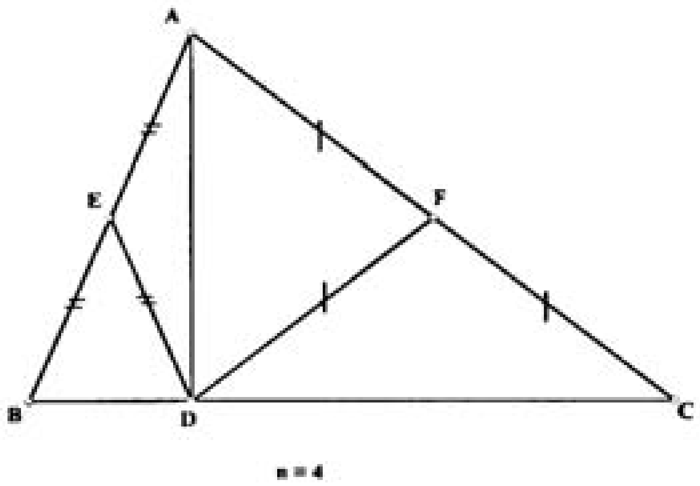


FIGURE 8

Moreover, if any sub-triangle of a decomposition is subdivided into four isosceles triangles, the number of triangles is increased by three; therefore, as in the warm-up problem on squares, a solution of the case of n triangles automatically carries with it the whole infinite family of cases $\{n, n+3, n+6, \ldots\}$. We need consider, then, only the cases of $n = 4, 5$, and 6, and since 4 is already done, only 5 and 6 remain.

But $n = 6$ is merely a corollary of $n = 4$. After drawing the interior altitude AD, thus partitioning ABC into two right-angled triangles, further subdivide one of these into two isosceles triangles by joining D to the midpoint of the

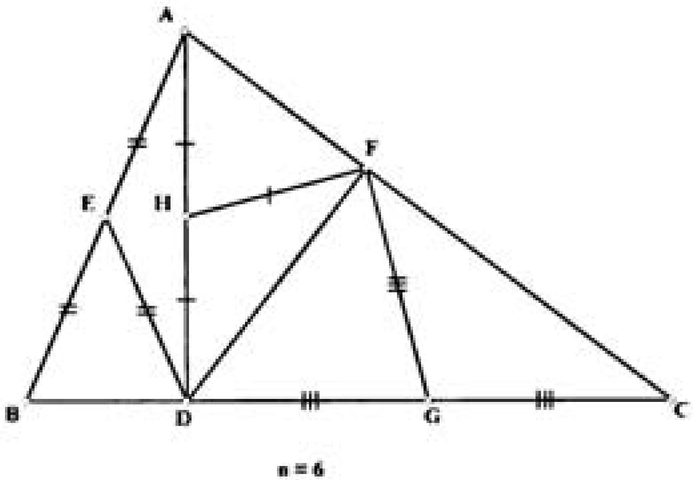


FIGURE 9

hypotenuse (say DE), and subdivide the other into four isosceles triangles as above.

For $n = 5$ it is convenient to consider separately the special case when the given triangle ABC is equilateral.

(i) In general, ABC is not equilateral and has a pair of unequal sides. If $AB < BC$, let $BD = BA$ be cut off along BC to give isosceles triangle ABD. Subdividing the remainder ADC into four isosceles triangles, then, we obtain the desired decomposition into five triangles.

(ii) Clearly this approach doesn't work when ABC is equilateral. Now, the circumcenter O of an equilateral triangle is also the orthocenter (where the altitudes meet), the incenter, and the centroid — you might say it's *the* center of the triangle. However, if one drops down the altitude AO a little closer to the

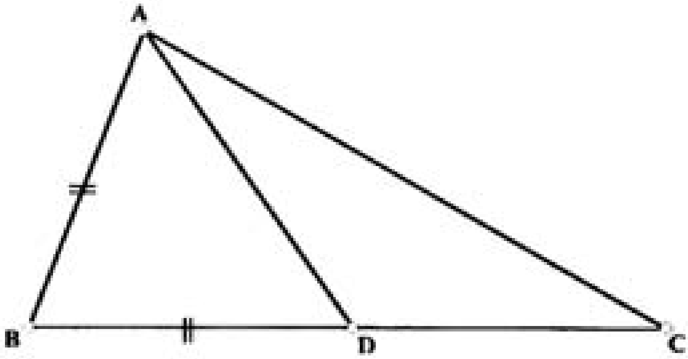


FIGURE 10

opposite side BC to a center D, and then draws a circle through B and C, the circle will no longer be big enough to reach the vertex A, but will cross AB and AC in points E and F which, because of the obvious symmetry, will cut off equal intercepts AE and AF, making $\triangle AEF$ isosceles (in fact equilateral). Thus EF, with DB, DC, DE, and DF, provide the required decomposition of $\triangle ABC$ into five isosceles triangles.

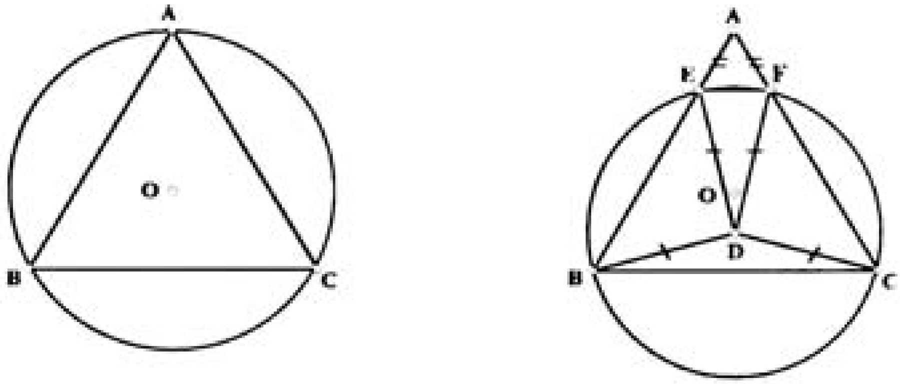


FIGURE 11

AIME—1987

Now let us look at three problems from the 1987 American Invitational Mathematics Examination (AIME).

Problem 1

It is not difficult to see that every odd positive integer $2k + 1$ is the sum of some string of consecutive positive integers. If nothing else, there is always the trivial $k + (k + 1) = 2k + 1$. But some odd integers sponsor several such strings; for example,

$$21 = 10 + 11 = 6 + 7 + 8 = 1 + 2 + 3 + 4 + 5 + 6.$$

In this problem we are asked to determine the length k of the longest string of consecutive positive integers that add up to 3^{11}.

Solution

If a longest such string of k consecutive positive integers begins at the integer a, we have

$$a + (a + 1) + \cdots + (a + k - 1) = 3^{11},$$

$$\frac{k}{2}(2a + k - 1) = 3^{11},$$

$$k(2a + k - 1) = 2 \cdot 3^{11},$$

which reveals that k must divide $2 \cdot 3^{11}$. Thus k must be one of the 24 numbers

$$\{1, 3, 3^2, \ldots, 3^{11}, 2, 2 \cdot 3, 2 \cdot 3^2, \ldots, 2 \cdot 3^{11}\}.$$

While it hardly seems worthwhile mentioning, it is certainly true that the first number in our string must be at least 1, making the sum at least

$$1 + 2 + \cdots + k = \tfrac{1}{2}k(k+1).$$

Thus, for whatever it's worth, we must have the sum $3^{11} \geq \frac{1}{2}k(k+1)$, giving

$$k(k+1) = k^2 + k \leq 2 \cdot 3^{11} = 354294.$$

At least we can conclude that k must be less than the square root of $2 \cdot 3^{11}$:

$$k < \sqrt{354294} = 595.226\ldots,$$

implying

$$k < 596.$$

It seems that this has some merit after all, for this eliminates fully half of the 24 possibilities, leaving just

$$k \in \{1, 3, 9, 27, 81, 243, 2, 6, 18, 54, 162, 486\}.$$

Since there doesn't seem to be any other bounding condition on the horizon, let's start checking these possibilities in the hope that one of the biggest of them will turn out to be feasible.

We have in all cases that

$$k(2a + k - 1) = 2 \cdot 3^{11},$$

and therefore the substitution $k = 486 = 2 \cdot 3^5$ would lead to

$$2a + 485 = \frac{2 \cdot 3^{11}}{2 \cdot 3^5} = 3^6 = 729,$$

$$2a = 244, \quad \text{and} \quad a = 122, \ \textit{a positive integer}.$$

On our very first try, then, we have found that $k = 486$ *is* feasible, and is clearly the greatest possible k. The corresponding series is

$$\begin{aligned} 122 + 123 + \cdots + 607 &= \tfrac{1}{2} \cdot 486(244 + 485) \\ &= \tfrac{1}{2}(2 \cdot 3^5)(729) = 3^5 \cdot 3^6 = 3^{11}. \end{aligned}$$

Problem 2

Next let us look at the problem I consider to be the highlight of the examination.

It concerns one of the early sorting procedures for arranging a set of numbers in increasing order, called a bubble sort. An arbitrary order of the numbers is likely to contain many "inversions," that is, a larger number ahead of a smaller one, and our algorithm is a primitive procedure for seeking out the inversions and undoing them, one at a time, until they are all gone.

Beginning with a given order $r_1, r_2, \ldots, r_n$, the inversions are sought out by comparing the numbers in the first two places, (r_1, r_2), then the pair in positions 2 and 3, followed by the pair in positions 3 and 4, and so on. Whenever an inversion is found, one simply undoes it and proceeds to the succeeding pair of positions. Such a "bubble pass" clearly does not skip over a larger number, but carries it along until it is displaced by an even bigger number. Thus, on the first pass, the greatest number is picked up somewhere along the way and carried right to the end; on the second pass, the next biggest number is carried to the second last place (next to the biggest number at the end), and so forth. Of course, many inversions are incidently undone along the way as the big numbers are piled up at the end.

To illustrate the basic idea, the first bubble pass over the set $\{1,9,8,7\}$ would produce the stages

$$\underline{1,9},8,7 \to 1,\underline{9,8},7 \to 1,8,\underline{9,7} \to 1,8,7,9;$$

and the second pass would complete the task:

$$\underline{1,8},7,9 \to 1,\underline{8,7},9 \to 1,7,\underline{8,9} \to 1,7,8,9.$$

Now for the problem.

> Suppose $r_1, r_2, \ldots, r_{40}$ is a random arrangement of 40 different numbers. What is the probability that r_{20} will be carried forward to end up in position 30 after the first bubble pass?

Solution

As we have noted, a bubble pass picks up a number and carries it forward, undoing inversions, until a bigger number is encountered. At each stage, the number "currently in position i" is compared with "the original occupant of position $i+1$", i.e., (current r_i, r_{i+1}), implying that the current r_i is always the biggest number encountered up to that point: in all cases,

$$\text{the current } r_i = \max\{r_1, r_2, \ldots, r_i\}.$$

The current r_i remains the reigning "largest number to date" until it is defeated in a comparison with a greater number. When this happens, the "current r_i" becomes the final resident in position i, i.e., the "final r_i," although it still retains the distinction of being the greatest of the numbers in the first i positions. The larger r_{i+1}, then, is clearly the greatest number in the first $i+1$ positions. Consequently, if we want r_{20} to move forward and end up as "final r_{30}," r_{20} must be the greatest number in the first 30 positions and r_{31} must be the greatest number in the first 31 positions. Our problem is to determine the probability that these two features occur in a random ordering of the 40 given numbers.

Fortunately, this is really an easy problem. Since there is no preference for one of the numbers over another, the biggest of the first i numbers will occur in a particular position the same number of times as it will occur in any other of the first i positions. Thus the probability of r_{20} being the greatest in the first 30 positions is $1/30$ and the probability of r_{31} being the biggest in the first 31 positions is $1/31$. Since these are independent contingencies, the probability of a favorable result is simply

$$\frac{1}{30}\cdot\frac{1}{31}=\frac{1}{930}.$$

Problem 3

In our third problem we are asked simply to calculate the value of

$$N=\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}$$

Solution

Surely there must be some way of factoring these expressions. Since $324=18^2$, we might well be reminded of the unfactorable forms a^2+b^2 and a^4+b^4. However, in the present case, each factor is in the mixed form a^4+18^2, which we should at least make an attempt to factor. Completing the square, we get

$$\begin{aligned} a^4+18^2 &= (a^2+18)^2-36a^2 \\ &= (a^2+18+6a)(a^2+18-6a), \end{aligned}$$

and so the given number N, which may be written as

$$N=\prod_{k=0}^{4}\frac{(10+12k)^4+18^2}{(4+12k)^4+18^2},$$

is given by

$$\prod_{k=0}^{4}\frac{\left[(10+12k)^2+18+6(10+12k)\right]\left[(10+12k)^2+18-6(10+12k)\right]}{\left[(4+12k)^2+18+6(4+12k)\right]\left[(4+12k)^2+18-6(4+12k)\right]}$$
$$=\prod_{k=0}^{4}\frac{(144k^2+312k+178)(144k^2+168k+58)}{(144k^2+168k+58)(144k^2+24k+10)}.$$

Cancelling the equal factors in the numerator and denominator gives

$$N=\prod_{k=0}^{4}\frac{144k^2+312k+178}{144k^2+24k+10},$$

which we may denote by

$$N=\prod_{k=0}^{4}\frac{n_k}{d_k}=\frac{n_0n_1n_2n_3n_4}{d_0d_1d_2d_3d_4}.$$

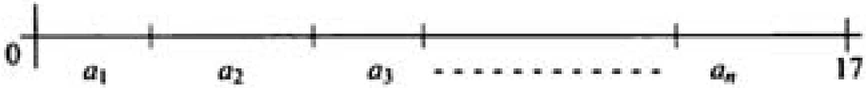
In these terms, it turns out that $d_{k+1}=n_k$:

$$d_{k+1}=144(k+1)^2+24(k+1)+10$$
$$=144k^2+312k+178=n_k,$$

and so N is just

$$\frac{n_4}{d_0}=\frac{144(4^2)+312(4)+178}{10}=\frac{3730}{10}=373.$$

A Problem from the 1991 AIME Examination



Clearly, for every integer $n > 1$, there is an infinite number of ways of partitioning the interval $(0, 17)$ into n nonempty parts $a_1, a_2, \ldots, a_n$ (any $n - 1$ different interior points will do it). Now, the sum

$$S_n(P) = \sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2}$$

$$= \sqrt{1^2 + a_1^2} + \sqrt{3^2 + a_2^2} + \cdots + \sqrt{(2n-1)^2 + a_n^2}$$

is a function of the partition $P = (a_1, a_2, \ldots, a_n)$ and assumes an infinity of values as P ranges over all possible partitions. Let S_n be the *minimum* value of this function:

$$S_n = \min_P S_n(P) = \min_P \sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2}.$$

Remarkably, it turns out that *exactly one* of the values

$$S_2, S_3, S_4, \ldots, S_n, \ldots$$

is an integer. Which one is it?

Solution

In view of the observation that $\sqrt{(2k-1)^2 + a_k^2}$ is the length of the hypotenuse of a right-triangle with sides $(2k-1)$ and a_k, making $S_n(P)$ the sum of n such hypotenuses, we might start thinking geometrically, and with this turn of mind, the problem is almost solved. It is a reasonably small step to thread together these hypotenuses to make a polygonal path from the origin to the point $(n^2, 17)$ in a coordinate plane.

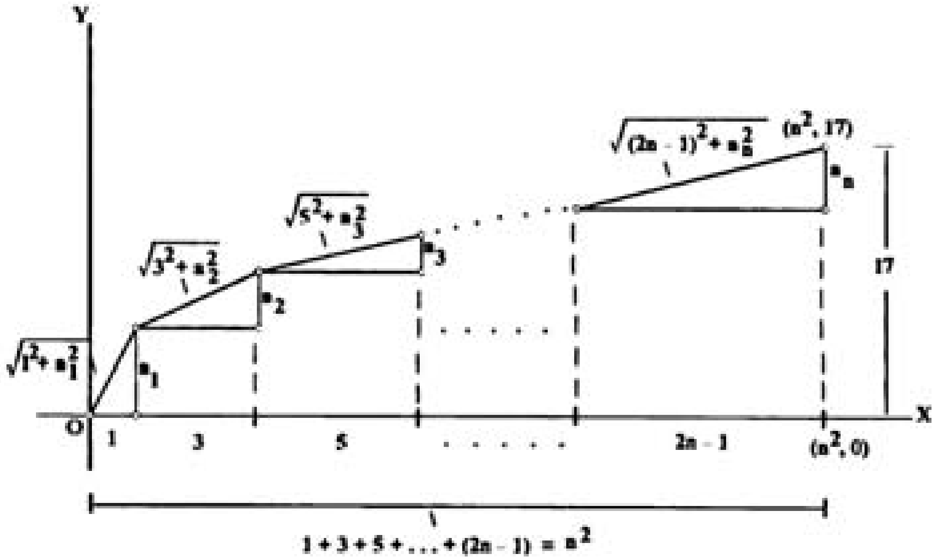


FIGURE 12

Clearly, then,

$$\begin{aligned} S_n &= \min S_n(P) \\ &= \text{the length of the } \textit{straight} \text{ path from} (0, 0) \text{ to } (n^2, 17) \\ &= \sqrt{n^4 + 17^2}. \end{aligned}$$

In order for S_n to be an integer t, we must have

$$n^4 + 17^2 = t^2,$$

making $(n^2, 17, t)$ a Pythagorean triple. Now, there is only one Pythagorean triple which has a leg of length 17. It is fairly well known, and in any case very

easy to verify that, if m is odd (like 17), then

$$\left(m, \tfrac{1}{2}(m^2 - 1), \tfrac{1}{2}(m^2 + 1)\right)$$

is a Pythagorean triple. Hence $(17, 144, 145)$ is the desired triple, giving

$$n^2 = 144 \quad \text{and} \quad n = 12.$$

Thus

$$S_{12} = 145 \quad \text{is the only integer among the } S_n.$$

Nine Unused Problems from the 1987 International Olympiad

1. From the USA

(*Crux Mathematicorum*, 1987, 279; a similar solution is given in 1989, 135.)

Consider the number

$$N = 025865413989732.$$

As it is scanned from the left, clearly the digits fall into six increasing and decreasing strings which alternate along the number:

0258, 86541, 139, 98, 89, and 9732.

$$\begin{array}{ccccccccccccccc}
 & & & 8 & & & & & & 9 & & 9 & & & \\
 & & 5 & & 6 & & & & & & 8 & & 7 & & \\
 & 2 & & & & 5 & & & 3 & & & & & 3 & \\
0 & & & & & & 4 & & & & & & & & 2 \\
 & & & & & & & 1 & & & & & & &
\end{array}$$

Not wishing to consider a long string as two or more abutting shorter ones, let us deal only with *maximal* strings, that is, ones which go all the way to the next change of direction. This eliminates any ambiguity as to what is and what is not a string under consideration. Henceforth, let us refer to such maximal strings simply as "runs."

Numbers like 77765589911, containing consecutively repeated digits, introduce unwanted complications, and so let us confine our attention to positive integers in which adjacent digits are always different. Finally, as in the first example, let us allow the leading digit to be 0.

The question is "What is the *average number of runs* in such an n-digit integer?"

Solution

The following beautiful solution is due to my colleague Ian Goulden (University of Waterloo).

Clearly the number of such n-digit integers is $N = 10 \cdot 9^{n-1}$: there are 10 choices for the first digit, but only 9 for each succeeding digit, since consecutive repetitions are forbidden. Thus the required average is

$$A = \frac{\text{the total number } T \text{ of runs in all } N \text{ integers}}{N} = \frac{T}{N}.$$

The problem is to get a line on T.

Since a run can generally begin anywhere along an integer, a simple-minded way of calculating T would be to add up

(the number r_1 of runs that begin at the first digit)
+(the number r_2 of runs that begin at the second digit)
+(the number r_3 of runs that begin at the third digit)
....
+(the number r_n of runs that begin at the nth digit),

that is,

$$T = r_1 + r_2 + \cdots + r_n.$$

In these terms,

$$\text{the average } A = \frac{r_1 + r_2 + \cdots + r_n}{N} = \sum_{i=1}^{n} \frac{r_i}{N}.$$

Clearly r_i/N is the probability that an integer chosen at random from our N numbers will be found to have a run that begins in the ith place. Denoting this probability by p_i, the desired average is simply

$$A = \sum_{i=1}^{n} p_i.$$

Therefore, let us turn our attention to the calculation of these probabilities p_i.

Clearly $p_1 = 1$, since the leading digit always begins a run, and $p_n = 0$, since the last digit never does. It's the interior places that are the challenge, but these p_i can be determined from the following ingenious approach.

In order to deal with any interior position i, *consider the position ahead of it and the one after it*. The following schematic figures show the only four ways the digits in three adjacent positions can comport themselves:

(1) (2) (3) (4)

Thus, the middle digit, in position i, begins a run only in cases (3) and (4), and the probability

$$p_i = \text{prob (3)} + \text{prob (4)}.$$

This quantity is a problem to calculate directly, but is easily obtained from the complementary point of view:

$$\begin{aligned} p_i &= 1 - [\text{prob (1)} + \text{prob (2)}] \\ &= 1 - 2 \text{ prob (1)}, \end{aligned}$$

since, by symmetry, it is obvious that cases (1) and (2) have the same probability.

Since the three digits in case (1) steadily increase, they must be one of the $\binom{10}{3}$ sets of three different digits (there is only one way to arrange them in increasing order). Now, the total number of ways three consecutive places can be filled is $10 \cdot 9^2$ (10 ways for the first place and 9 for each of the other two), and so

$$p_i = 1 - 2 \cdot \frac{\binom{10}{3}}{10 \cdot 9^2} = 1 - \frac{2 \cdot 10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 10 \cdot 9 \cdot 9} = 1 - \frac{8}{27} = \frac{19}{27}.$$

This comes as quite a surprise to me, for I wouldn't have thought that p_i would be a constant for all $n - 2$ interior positions and for all values of n. In any case, the required average is therefore

$$\begin{aligned} A &= \sum_{i=1}^{n} p_i = 1 + (n-2)p_i + 0 \\ &= 1 + \frac{19}{27}(n-2). \end{aligned}$$

Comments

In a 29-digit integer, then, there is an average of 20 runs. To a statistically untutored intuition like mine, this appeared to be quite high, for it seemed to

imply an extremely short *average length* of

$$\frac{29}{20} = 1 \cdot 45 \quad \text{digits per run.}$$

Since no run can contain 1·45 digits and not all runs can be longer than average, this meant that some under-average run must contain only a single digit, an impossible situation. Of course, this impetuous calculation neglects the fact that the 20 runs are hooked together at 19 places which count double toward the lengths of the runs, since each is both the end of one run and the beginning of the next. Thus the average length of a run is actually

$$\frac{29+19}{20} = \frac{48}{20} = 2 \cdot 4,$$

which is certainly feasible, but which still struck me as none too big.

Checking 56-digit integers, the average number of runs is

$$1 + \frac{19}{27}(54) = 39,$$

giving an average length of

$$\frac{56+38}{39} = \frac{94}{39} = 2 \cdot 410256\ldots.$$

With such precious little improvement as this, there seemed little point in not going right to the general case. Accordingly, the average length of a run in an n-digit integer is

$$\frac{n + \frac{19}{27}(n-2)}{1 + \frac{19}{27}(n-2)} = \frac{48n - 38}{19n - 11} = \frac{48 - \frac{38}{n}}{19 - \frac{11}{n}},$$

which approaches the discouragingly small limit of $\frac{48}{19} = 2.5263\ldots$.

Perhaps the problem is that thinking generally in terms of zig-zag figures does not take into account the fact that no run can exceed a length of 10 digits. Surely this cramps things and plays a significant role in holding down the average length of a run. Of course, the upper limit on the length of a run can be eased by going to a larger number-base.

In base 101, for example, we would have

$$p_i = 1 - 2 \cdot \frac{\binom{101}{3}}{101.100^2} = 1 - \frac{2 \cdot 101 \cdot 100 \cdot 99}{3 \cdot 2 \cdot 101 \cdot 100 \cdot 100}$$

$$= 1 - \frac{99}{3 \cdot 100} = 1 - \frac{33}{100} = \frac{67}{100},$$

and the average number of runs in a 1002-digit integer would be

$$1 + \frac{67}{100}(1000) = 671.$$

Thus the average length of a run in such an integer would be

$$\frac{1002 + 670}{671} = \frac{1672}{671} = 2 \cdot 4918 \ldots,$$

which only goes to show that the state of my intuition on such matters is even worse than I thought it was. At the risk of destroying what little confidence I have left, let's take a brief look at lengthy integers in large bases b. We have

$$\begin{aligned} p_i &= 1 - 2 \cdot \frac{\binom{b}{3}}{b(b-1)^2} = 1 - \frac{2b(b-1)(b-2)}{3 \cdot 2 \cdot b(b-1)(b-1)} \\ &= 1 - \frac{b-2}{3(b-1)} = 1 - \frac{1}{3}\left(1 - \frac{1}{b-1}\right) \\ &= \frac{2}{3} + \frac{1}{3(b-1)}, \end{aligned}$$

a value that decreases to $\frac{2}{3}$ as b increases. For large b, then, the average number of runs is approximately $1 + \frac{2}{3}(n-2)$, giving an average length of

$$\frac{n + \frac{2}{3}(n-2)}{1 + \frac{2}{3}(n-2)} = \frac{5n-4}{2n-1} = \frac{5 - \frac{4}{n}}{2 - \frac{1}{n}},$$

which is never very far away from the magic number $2\frac{1}{2}$.

2. From Great Britain

(*Crux Mathematicorum*, 1987, 245; a similar solution is given in 1989, 9)

The subject of this splendid problem is a certain family of infinite sequences of *positive* real numbers

$$\{x_i\} = \{x_1, x_2, x_3, \ldots\}.$$

For the sequences in the family, we are concerned with the values of the fraction

$$f_n = \frac{\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n}}{\sqrt{x_1 + x_2 + \cdots + x_n}}$$

as n runs through the positive integers. For an arbitrary sequence, f_n can get indefinitely large. For example, $\{1, 1, 1, \ldots\}$ gives

$$f_n = \frac{1 + 1 + \cdots + 1}{\sqrt{1 + 1 + \cdots + 1}} = \frac{n}{\sqrt{n}} = \sqrt{n}, \text{ which goes to } \infty \text{ with } n;$$

a less trivial example is $\{1^2, 2^2, 3^2, \ldots, n^2, \ldots\}$.

However, if you stick to sequences in which each term is always at least as big as all the previous terms put together,

$$x_n \geq x_1 + x_2 + \cdots + x_{n-1} \quad \text{for all} \quad n = 2, 3, 4, \ldots,$$

then f_n does not get out of hand. In fact, f_n never gets any bigger than a certain fixed number c, no matter what the value of n nor which sequence of this kind you might choose to deal with. In this problem we are asked to determine this least upper bound c of all the f_n generated by this special family of sequences.

Solution

If the value of c had been disclosed in the problem, we would still have our work cut out for us in showing that it is in fact the least upper bound of the values of f_n. Since c is not given, we are saddled with the additional burden of speculating on its value before getting down to what would appear to be the major problem of establishing its character. Obviously, checking out any particular conjecture holds the alarming prospect of possibly being a complete waste of time. This certainly adds a little spice to the adventure, and most assuredly encourages us to put our faith only in highly promising possibilities.

The given condition, $x_n \geq x_1 + x_2 + \cdots + x_{n-1}$, allows a great deal of slack in the values of x_n. It seems feasible that the gaps between the actual values of x_n and their minimum possible values of $x_1 + \cdots + x_{n-1}$ might have a critical bearing on the values of f_n. It is not inconceivable, then, that a sequence in which these gaps are extreme values might correspond to an extreme set of f_n and lead us to the exact value of c. Accordingly, let's consider a completely "tightened" sequence in which, after x_1, each x_n is as small as possible, that is, x_n is always *equal to* $x_1 + \cdots + x_{n-1}$; for example,

$$S = \{1, 1, 2, 4, 8, 16, \ldots, 2^k, \ldots\}$$

$$(\text{recall } 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} = 2^k - 1).$$

For this sequence S,

$$f_n = \frac{\sqrt{1} + \sqrt{1} + \sqrt{2} + \sqrt{4} + \sqrt{8} + \cdots + \sqrt{2^{n-2}}}{\sqrt{1 + 1 + 2 + 4 + 8 + \cdots + 2^{n-2}}}$$

that is,

$$f_n = \frac{1 + 1 + \sqrt{2} + \sqrt{4} + \cdots + \sqrt{2^{n-2}}}{\sqrt{2^{n-1}}}.$$

For n odd, this gives

$$\begin{aligned}
f_{2k+1} &= \frac{1 + 1 + \sqrt{2} + \sqrt{4} + \sqrt{8} + \sqrt{16} + \cdots + \sqrt{2^{2k-2}} + \sqrt{2^{2k-1}}}{\sqrt{2^{2k}}} \\
&= \frac{1 + (1 + 2 + 4 + \cdots + 2^{k-1}) + \sqrt{2}(1 + 2 + 4 + \cdots + 2^{k-1})}{2^k} \\
&= \frac{1 + (2^k - 1) + \sqrt{2}(2^k - 1)}{2^k} \\
&= 1 + \sqrt{2}\left(1 - \frac{1}{2^k}\right).
\end{aligned}$$

Therefore, as $k \to \infty$, $f_{2k+1} \to 1 + \sqrt{2}$.

For n even, we have

$$\begin{aligned}
f_{2k+2} &= \frac{1 + 1 + \sqrt{2} + \cdots + \sqrt{2^{2k-1}} + \sqrt{2^{2k}}}{\sqrt{2^{2k+1}}} \\
&= \frac{1 + (1 + 2 + \cdots + 2^k) + \sqrt{2}(1 + 2 + \cdots + 2^{k-1})}{2^k\sqrt{2}} \\
&= \frac{1 + (2^{k+1} - 1) + \sqrt{2}(2^k - 1)}{2^k\sqrt{2}} \\
&= \sqrt{2} + \left(1 - \frac{1}{2^k}\right),
\end{aligned}$$

and again

$$f_{2k+2} \to \sqrt{2} + 1 \quad \text{as} \quad k \to \infty.$$

Thus the conjecture $c = 1 + \sqrt{2}$ is far from a wild guess and, coming from an extreme sequence, seems worthy of a serious attempt to establish its validity. We know that f_n takes values that are arbitrarily close to $1 + \sqrt{2}$, and if we could show that f_n never exceeds $1 + \sqrt{2}$, the desired conclusion would follow.

Clearly,

$$f_1 = \frac{\sqrt{x_1}}{\sqrt{x_1}} = 1 \le 1 + \sqrt{2}.$$

and induction would appear to be a reasonable way to approach the problem. Accordingly, suppose

$$f_{n-1} = \frac{\sqrt{x_1} + \cdots + \sqrt{x_{n-1}}}{\sqrt{x_1 + \cdots + x_{n-1}}} \le 1 + \sqrt{2}, \tag{A}$$

and let us try to show that

$$f_n = \frac{\sqrt{x_1} + \cdots + \sqrt{x_n}}{\sqrt{x_1 + \cdots + x_n}}$$

is also $\le 1 + \sqrt{2}$. From (A), we can get a bounded substitution for either of the expressions

$$\sqrt{x_1} + \cdots + \sqrt{x_{n-1}} \quad \text{or} \quad x_1 + x_2 + \cdots + x_{n-1}.$$

Although appearing to imply undesirable complications, let us choose

$$x_1 + x_2 + \cdots + x_{n-1} \ge \left(\frac{\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_{n-1}}}{1 + \sqrt{2}} \right)^2,$$

from which we obtain

$$\begin{aligned} f_n &\le \frac{\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n}}{\sqrt{\left[\dfrac{\sqrt{x_1} + \cdots + \sqrt{x_{n-1}}}{1 + \sqrt{2}}\right]^2 + x_n}} \\ &= \frac{(1 + \sqrt{2})(\sqrt{x_1} + \cdots + \sqrt{x_n})}{\sqrt{(\sqrt{x_1} + \cdots + \sqrt{x_{n-1}})^2 + (1 + \sqrt{2})^2 x_n}}. \end{aligned}$$

Thus the desired conclusion, $f_n \le 1 + \sqrt{2}$, would follow if we could show that

$$\frac{\sqrt{x_1} + \cdots + \sqrt{x_n}}{\sqrt{(\sqrt{x_1} + \cdots + \sqrt{x_{n-1}})^2 + (1 + \sqrt{2})^2 x_n}} \le 1,$$

that is,

$$(\sqrt{x_1} + \cdots + \sqrt{x_{n-1}}) + \sqrt{x_n} \le \sqrt{(\sqrt{x_1} + \cdots + \sqrt{x_{n-1}})^2 + (1 + \sqrt{2})^2 x_n}.$$

Squaring shows this is equivalent to

$$\begin{aligned} &(\sqrt{x_1} + \cdots + \sqrt{x_{n-1}})^2 + 2\sqrt{x_n}(\sqrt{x_1} + \cdots + \sqrt{x_{n-1}}) + x_n \\ &\quad \le (\sqrt{x_1} + \cdots + \sqrt{x_{n-1}})^2 + (3 + 2\sqrt{2})x_n, \end{aligned}$$

and

$$2\sqrt{x_n}(\sqrt{x_1} + \cdots + \sqrt{x_{n-1}}) \le (2 + 2\sqrt{2})x_n,$$
$$\sqrt{x_1} + \cdots + \sqrt{x_{n-1}} \le (1 + \sqrt{2})\sqrt{x_n}.$$

But

$$x_n \ge x_1 + x_2 + \cdots + x_{n-1},$$

and so

$$\sqrt{x_n} \ge \sqrt{x_1 + x_2 + \cdots + x_{n-1}} \qquad \text{(B)}$$

Now, from the induction hypothesis (A), we have

$$\sqrt{x_1} + \cdots + \sqrt{x_{n-1}} \le (1 + \sqrt{2})\sqrt{x_1 + \cdots + x_{n-1}}.$$

Replacing $\sqrt{x_1 + \cdots + x_{n-1}}$ in this by $\sqrt{x_n}$, then, gives the desired

$$\sqrt{x_1} + \cdots + \sqrt{x_{n-1}} \le (1 + \sqrt{2})\sqrt{x_n}$$

by (B).

Since all these steps are reversible, it follows by induction that f_n is less than or equal to $1 + \sqrt{2}$ for all n and all sequences; and since f_n gets arbitrarily close to $1 + \sqrt{2}$, as demonstrated for the sequence $\{1, 1, 2, 4, 8, \ldots, 2^k, \ldots\}$, we conclude that the least upper bound c is indeed equal to $1 + \sqrt{2}$.

3. From Iceland

(*Crux Mathematicorum*, 1987, 278; an alternative solution is given in 1989, 133)

Five different numbers are drawn at random from $\{1, 2, \ldots, n\}$, one at a time, without replacement. Show that the probability that the first three numbers drawn, as well as all five numbers, can be arranged to form an arithmetic progression is greater than

$$\frac{6}{(n-2)^3}.$$

Solution

First of all, let's count the number of 5-term arithmetic progressions in which the common difference is the positive integer d. If the first term is a, then the fifth

term, $a+4d$, must not exceed n, i.e.,

$$a+4d \le n, \quad \text{and} \quad a \le n-4d.$$

That is to say, the first term can be any of the integers $1, 2, \ldots, n-4d$, and so the number of such progressions is simply $n-4d$.

Next, let's investigate the possible values of d. The condition $a+4d < n$ also gives

$$d \le \frac{n-a}{4},$$

and since a must be at least 1, we have

$$d \le \frac{n-1}{4}$$

in all cases. Because d is an integer, this yields

$$d \le \left[\frac{n-1}{4}\right],$$

where the square brackets indicate integer part.

Just to be on the safe side, we had better check whether this largest value $d = \left[\frac{n-1}{4}\right]$ is always possible. For $a = 1$, we have

$$1 + 4\left[\frac{(n-1)}{4}\right] \le 1 + 4 \cdot \frac{n-1}{4} = n,$$

and it is clear that d can be any of the integers

$$1, 2, 3, \ldots, \left[\frac{n-1}{4}\right].$$

Thus the total number of 5-term arithmetic progressions $(t_1, t_2, t_3, t_4, t_5)$ in $\{1, 2, \ldots, n\}$ is

$$\begin{aligned} S &= \sum_{d=1}^{\left[\frac{n-1}{4}\right]} (n-4d) \\ &= n\left[\frac{n-1}{4}\right] - 4\left(1 + 2 + \cdots + \left[\frac{n-1}{4}\right]\right) \\ &= n\left[\frac{n-1}{4}\right] - 2\left[\frac{n-1}{4}\right]\left(\left[\frac{n-1}{4}\right] + 1\right) \\ &= \left[\frac{n-1}{4}\right]\left(n - 2\left[\frac{n-1}{4}\right] - 2\right). \end{aligned}$$

Now, when the numbers drawn can be arranged into a 5-term arithmetic progression $(t_1, t_2, t_3, t_4, t_5)$, in order to have the first three that are drawn also capable of being arranged into an arithmetic progression, these three numbers would have to constitute one of the four sets

$$\{t_1, t_2, t_3\},\quad \{t_2, t_3, t_4\},\quad \{t_3, t_4, t_5\},\quad \{t_1, t_3, t_5\}.$$

The set $\{t_1, t_2, t_3\}$, for example, could be drawn in any of its $3! = 6$ possible orders, and the final two numbers, t_4 and t_5, could occur in either order, and so there are $6 \cdot 2 = 12$ different ways of drawing the set $\{t_1, t_2, t_3, t_4, t_5\}$ in which the first three numbers drawn are t_1, t_2, t_3, in some order. Similarly for each of the four cases, for a total of $4 \cdot 12 = 48$ orders in which each of the S progressions can be drawn so that the first three numbers can also be arranged in arithmetic progression. Thus, of all the $n(n-1)(n-2)(n-3)(n-4)$ ways of drawing five numbers from $\{1, 2, \ldots, n\}$, there are $48S$ favorable cases, for a probability

$$P = \frac{48S}{n(n-1)(n-2)(n-3)(n-4)}.$$

It remains to estimate the value of S.

In dropping the fractional part of $\dfrac{n-1}{4}$ in order to get $\left[\dfrac{n-1}{4}\right]$, the loss incurred is either $\dfrac{0}{4}, \dfrac{1}{4}, \dfrac{2}{4}$, or $\dfrac{3}{4}$. Hence $\left[\dfrac{n-1}{4}\right]$ is one of the four values

$$\frac{n-1}{4},\quad \frac{n-2}{4},\quad \frac{n-3}{4},\quad \text{or}\quad \frac{n-4}{4}.$$

The corresponding values of S are

(a) $$\frac{(n-1)}{4}\left(n-2\left(\frac{n-1}{4}\right)-2\right) = \frac{n^2-n}{4} - \frac{n^2-2n+1}{8} - \frac{n-1}{2}$$
$$= \frac{n^2-4n+3}{8};$$

(b) $$\frac{(n-2)}{4}\left(n-2\left(\frac{n-2}{4}\right)-2\right) = \frac{n^2-2n}{4} - \frac{n^2-4n+4}{8} - \frac{n-2}{2}$$
$$= \frac{n^2-4n+4}{8};$$

(c) $$\frac{n-3}{4}\left(n-2\left(\frac{n-3}{4}\right)-2\right) = \frac{n^2-3n}{4} - \frac{n^2-6n+9}{8} - \frac{n-3}{2}$$
$$= \frac{n^2-4n+3}{8};$$

(d) $$\frac{n-4}{4}\left(n-2\left(\frac{n-4}{4}\right)-2\right)=\frac{n^2-4n}{4}-\frac{n^2-8n+16}{8}-\frac{n-4}{2}$$
$$=\frac{n^2-4n}{8},$$

the *smallest* of which is

$$\frac{n^2-4n}{8}=\frac{n(n-4)}{8}.$$

Thus, in all cases,

$$S\geq\frac{n(n-4)}{8},$$

and so

$$P\geq\frac{48\cdot\dfrac{n(n-4)}{8}}{n(n-1)(n-2)(n-3)(n-4)}$$
$$=\frac{6}{(n-1)(n-2)(n-3)}.$$

Now,

$$\begin{aligned}(n-1)(n-3)&=n^2-4n+3\\&<n^2-4n+4\\&=(n-2)^2,\end{aligned}$$

implying

$$\frac{1}{(n-1)(n-3)}>\frac{1}{(n-2)^2},$$

and so

$$P\geq\frac{6}{(n-1)(n-2)(n-3)}$$
$$>\frac{6}{(n-2)^3},\quad\text{as desired.}$$

4. From Russia

(*Crux Mathematicorum*, 1987, 309; an alternative solution is given in 1989, 168).

Let $\tau(n)$ denote the *number* of positive divisors of the positive integer n. Then

$$\tau(1) = 1, \quad \tau(2) = 2, \quad \tau(3) = 2, \quad \tau(4) = 3, \ldots.$$

The subject of this problem is the family of sequences

$$n, \ \tau(n), \ \tau(\tau(n)), \ \tau(\tau(\tau(n))), \ldots,$$

in which each term after the first is the number of divisors of the preceding term. For example,

$$540, 24, 8, 4, \ldots.$$

Obviously, such a sequence is completely determined by its first term, which can be any positive integer $n > 1$.

The problem is to identify the sequences in this family which do not contain a perfect square.

Solution

If the prime decomposition of n is

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

it is easy to see that

$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1);$$

any integer of the form

$$d = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$$

will be a divisor of n if and only if none of its exponents b_i exceed the corresponding exponent a_i in n, that is, iff

$$0 \le b_i \le a_i.$$

Thus there are $(a_1 + 1)$ choices for b_1, $(a_2 + 1)$ for b_2, etc., and since the b_i are independent of one another, the formula follows.

Now, $\tau(n)$ will be *odd* if and only if each of its factors $(a_i + 1)$ is odd, that is, if and only if each a_i is *even*. Hence we have the important result that $\tau(n)$ is odd if and only if n is a perfect square. Thus, if any term of one of our sequences is odd, the previous term will be a dreaded perfect square. Of course, the very first term doesn't have a previous term, and so it is permissible to begin a sequence with an

odd number. Outside of that, however, *to avoid perfect squares is to avoid odd numbers.*

Now let us establish the important property that $\tau(n)$ is a *decreasing* function for $n > 2$:

$$\tau(n) < n \quad \text{for} \quad n > 2.$$

If d is a divisor of n, it divides into n some number of times k, and we have

$$dk = n.$$

Then k is also a divisor of n, and (d, k) is a pair of complementary divisors. Now, clearly not both d and k can exceed $\sqrt{n}$, for then dk would be $> n$; on the other hand, they can't both be less than $\sqrt{n}$ either, for then $dk < n$. Thus a pair of unequal complementary divisors has no choice but to straddle $\sqrt{n}$. Of course there is also the possibility of a self-complementary integral divisor $\sqrt{n}$ itself, in the case of n being a perfect square. (We might observe in passing that, since the divisors generally pair up, with a possible extra divisor $\sqrt{n}$, this gives another proof of the fact that $\tau(n)$ is odd if and only if n is a square.) Now, since each complementary pair must contain an integer not exceeding $\sqrt{n}$, there can't be more than $[\sqrt{n}]$ of them (i.e., the integer part of $\sqrt{n}$), counting the possible self-complementary pair $(\sqrt{n}, \sqrt{n})$, and it follows, for all n, that

$$\tau(n) \leq 2[\sqrt{n}].$$

Now, if n is a square, then $\sqrt{n}$ *is* a divisor, but the pair $(\sqrt{n}, \sqrt{n})$ contributes only 1 to $\tau(n)$ instead of 2, making

$$\tau(n) \leq 2[\sqrt{n}] - 1 < 2\sqrt{n}.$$

If n is not a square, then $\sqrt{n}$ is not an integer, implying $[\sqrt{n}] < \sqrt{n}$, and again

$$\tau(n) \leq 2[\sqrt{n}] < 2\sqrt{n}.$$

Therefore $\tau(n) < 2\sqrt{n}$ for all n.

Now, for $n > 4$, clearly $4n < n \cdot n$ and $2\sqrt{n} < n$, giving $\tau(n) < n$. But $\tau(n) < n$ also holds for $n = 3$ and 4 ($\tau(3) = 2 < 3$ and $\tau(4) = 3 < 4$), and we have finally that

$$\tau(n) < n \quad \text{for all} \quad n > 2.$$

Thus the sequence $n, \tau(n), \tau(\tau(n)), \ldots$ is *strictly decreasing* so long as the terms remain greater than 2. But no term in a sequence is smaller than 2. The only positive integer with a single divisor is 1, and so a 1 can arise only if the previous term is also 1, implying that the only sequence with a 1 is $(1, 1, 1, \ldots)$, which starts that way. Since our sequences begin with $n > 1$, their smallest possible

entry is 2, and since $\tau(n)$ is a decreasing function for $n > 2$, we obtain the crucial property that every sequence must eventually decrease to the integer 2, after which it repeats indefinitely, since $\tau(2) = 2$.

Now, clearly $\tau(n) = 2$ if and only if n is a *prime number*. Therefore, sequences that begin with a prime p do not have a chance to produce a perfect square:

$$p, 2, 2, 2, 2, 2, 2, \ldots.$$

However, as we shall see, these are the only ones.

Suppose, then, that the first term in a sequence is a composite integer n. Thus $n \geq 4$, and, not being a prime, $\tau(n)$ cannot be 2, and so the endless round of 2's doesn't begin until at least the third term. As we have seen, the only way a 2 can arise is if the previous term is a prime. Of course, the previous term could be another 2, but the term immediately preceding the *first* 2 would have to be an *odd* prime. Being odd, the term ahead of this prime can be nothing else but a perfect square. Thus there is no avoiding a perfect square in any sequence which takes three or more terms to reach the first 2.

..., a perfect square, an odd prime, *the first* 2, 2, 2, 2, ...

Since prime values of n are the only ones to get the 2's started earlier, they determine the only square-free sequences in the family.

5. From France

(*Crux Mathematicorum*, 1987, 246)

Sometimes the presentation of a problem cheerfully leads us down the garden path. For example, consider the following problem.

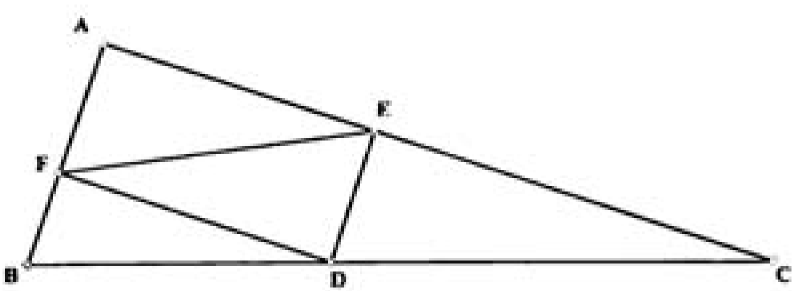


FIGURE 13

From a point D on the hypotenuse of BC of right triangle ABC, perpendiculars DE and DF are drawn to AC and AB, respectively. Determine the position of D for which EF has minimum length.

Of course, $AFDE$ is a rectangle, and it's not EF but the *other* diagonal AD that we should be thinking about. Obviously, AD has minimum length when D is at the foot of the altitude from A, showing how trivial the problem really is.

In this example, everything hinges on the fact that $\measuredangle A$ is a right angle, for then the diagonals EF and AD are always equal. It's not so simple when $\measuredangle A$ is not a right angle. Perhaps at this point you are sufficiently misdirected by this example for me to present the unused problem proposed by France.

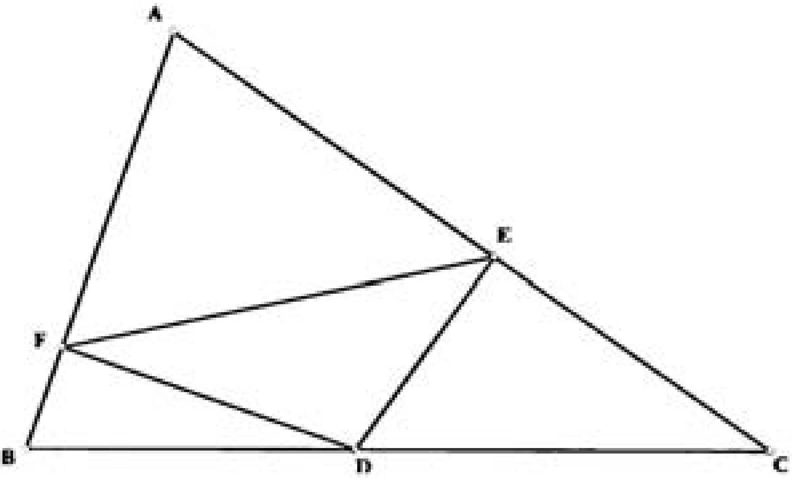


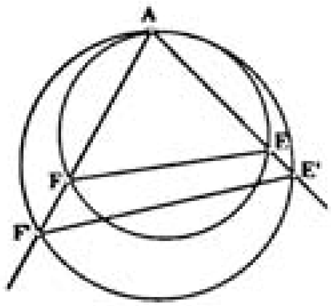
FIGURE 14

Where should D be placed to minimize EF for an arbitrary $\triangle ABC$?

Solution

Surprisingly, the answer is still the same—D should go at the foot of the altitude from A; and again it's because that's when AD is shortest. However, the connection between EF and AD is not quite as obvious this time. In the example, the important point is not really that $AFDE$ determines a rectangle, but that $AFDE$ is always *cyclic*. For each position of D on BC, EF subtends the *same angle* at the circumference of the circumcircle, namely $\measuredangle A$. But, clearly, the smaller the circle, the smaller the chord EF (Figure 15). Since the right angles at E and F make AD a diameter, the circle is smallest when AD is smallest.

If D is not allowed to move on BC beyond B or C and if angle B or C is obtuse, then this argument is still valid, but in this case the smallest circle is obtained when D is at the vertex of the obtuse angle.



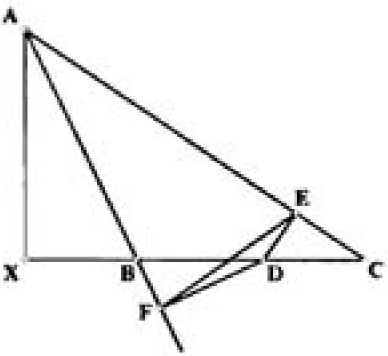


FIGURE 15

6. From Bulgaria

(*Crux Mathematicorum*, 1987, 309)

Suppose n points are chosen anywhere in the plane and segments are inserted between certain pairs of them so that, no matter which 4 of the points may be selected, the segments connecting some 3 of the 4 points form a triangle. What is the smallest number of segments that must be inserted to achieve this goal, and how should one choose which segments should be inserted in the minimum case?

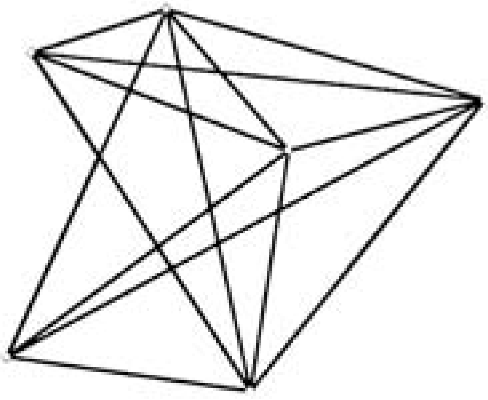


FIGURE 16

Solution

Clearly the resulting configuration may be viewed as a graph in which the n given points are the vertices and the inserted segments are the edges, and so it might be rewarding to think in terms of some of the simple concepts of graph theory. Recall that the number of edges at a vertex V is called the degree of v and is denoted by $d(v)$. Also, since each edge has two endpoints, each of which contributes 1 to the degree of its end-vertices, it follows that the sum of the degrees of all the vertices merely counts up all the endpoints and is therefore just twice the number of edges e:

$$\sum d(v) = 2e.$$

After a certain amount of experimentation, it appeared that the best one can do in accommodating this triangle requirement was to leave one vertex X entirely alone and put in every possible edge between the other $n - 1$ vertices R. This calls for $\binom{n-1}{2} = \frac{1}{2}(n-1)(n-2)$ edges and clearly does achieve the desired state:

> each 3 of a set 4 vertices from R form a triangle and, if X is among the 4 vertices selected, the other 3 come from R and thus determine a triangle.

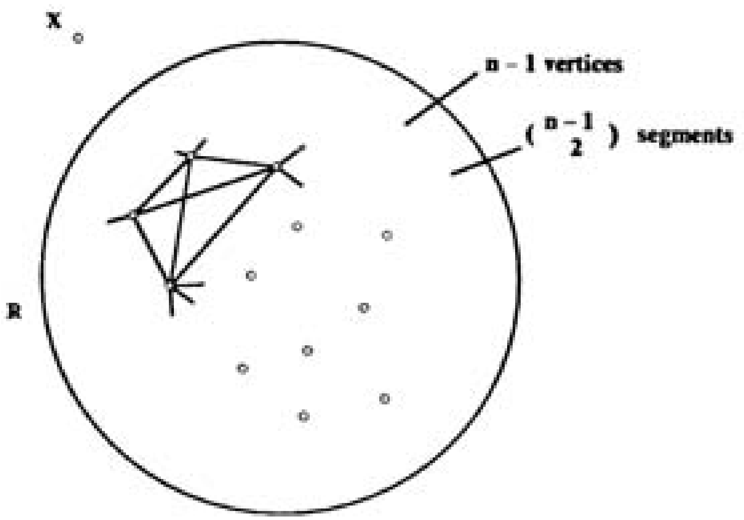


FIGURE 17

Indeed, if this is a minimum case, the question of which edges to insert reduces to a triviality. We would like to show that you can't get away with fewer than $\binom{n-1}{2}$ edges. Accordingly, let's try to obtain a contradiction on the assumption that not more than $\binom{n-1}{2} - 1$ edges are inserted. If

$$e \le \binom{n-1}{2} - 1 = \tfrac{1}{2}(n-1)(n-2) - 1,$$

then the sum of all the degrees would be

$$\sum d(v) = 2e \le (n-1)(n-2) - 2 = n^2 - 3n,$$

and the *average degree* $\le \frac{1}{n}(n^2 - 3n) = n - 3$.

Since not all degrees can be above average, some vertex v must have

$$d(v) \le n - 3,$$

in which case v is not connected to at least 2 other vertices x and y. Even if v were joined to all $n - 3$ vertices other than itself, x, and y, it is easy to obtain the contradiction that at least $\binom{n-1}{2}$ edges must be in the graph in order to insure that each 4 vertices contain some 3 that form a triangle.

Let R denote the set of $n - 1$ vertices other than v and consider the consequence that the edge between some two vertices of R is missing. There are three simple cases:

(i) the edge xy is missing,

(ii) an edge xz is missing at x (or equivalently at y),

(iii) an edge wz, with neither end at x or y, is missing.

Clearly, in (i), any fourth vertex Z gives a quadruple (v, x, y, z) that fails to contain a triangle, as does (v, x, y, z) in (ii) and (v, x, z, w) in (iii). Thus no edge

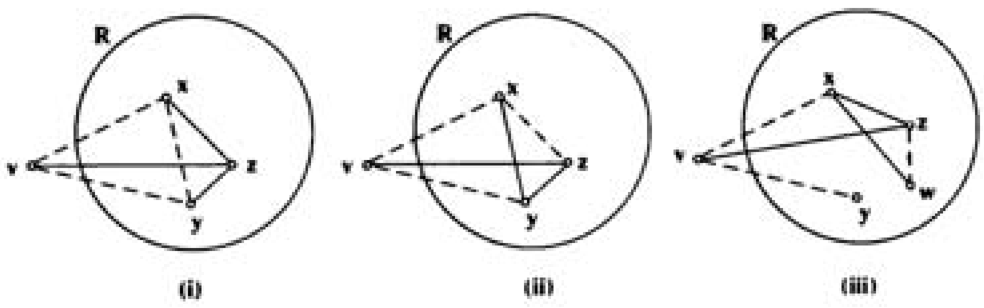


FIGURE 18

can be missing from R, implying all $\binom{n-1}{2}$ of its possible edges must be present, giving the desired contradiction.

7. From West Germany

Now let us look at an intriguing combinatorics problem that was proposed by West Germany.

How many n-digit integers are there, consisting only of the digits 1, 2, 3, 4, 5, in which adjacent digits always differ by exactly 1?

Solution

It is always wise to begin a problem like this by taking a detailed look at a few initial cases: for

$n = 1$: the integers are $\{1, 2, 3, 4, 5\}$, for a total of 5;
$n = 2$: we have $\{12, 21, 23, 32, 34, 43, 45, 54\}$, a total of 8;
$n = 3$: we have $\{121, 123, 212, 232, 234, 321, 323, 343, 345, 432, 434, 454, 543, 545\}$, a total of 14.

Letting a_n denote the number of integers of length n and $a(n, i)$ the number of those which end in the digit i, it is clear that

$$\begin{aligned}
a(n,1) &= a(n-1,2)\\
a(n,2) &= a(n-1,1) + a(n-1,3)\\
a(n,3) &= a(n-1,2) + a(n-1,4)\\
a(n,4) &= a(n-1,3) + a(n-1,5)\\
a(n,5) &= a(n-1,4).
\end{aligned}$$

With the initial values found from the cases $n = 1, 2, 3$, above, the following table may be continued as far as we like.

n	1	2	3	4	5	6	7	$\cdots$	$2k+1$	$2k+2$	$2k+3$	$2k+4$
$a(n,1)$	1	1	2	3	6	9	18	$\cdots$	$2\cdot 3^{k-1}$	3^k	$2\cdot 3^k$	3^{k+1}
$a(n,2)$	1	2	3	6	9	18	27	$\cdots$	3^k	$2\cdot 3^k$	3^{k+1}	$2\cdot 3^{k+1}$
$a(n,3)$	1	2	4	6	12	18	36	$\cdots$	$4\cdot 3^{k-1}$	$2\cdot 3^k$	$4\cdot 3^k$	$2\cdot 3^{k+1}$
$a(n,4)$	1	2	3	6	9	18	27	$\cdots$	3^k	$2\cdot 3^k$	3^{k+1}	$2\cdot 3^{k+1}$
$a(n,5)$	1	1	2	3	6	9	18	$\cdots$	$2\cdot 3^{k-1}$	3^k	$2\cdot 3^k$	3^{k+1}
a_n	5	8	14	24	42	72	126	$\cdots$				

Clearly there is a symmetry making $a(n,1) = a(n,5)$, and $a(n,2) = a(n,4)$. With $a_1 = 5$, $a_3 = 14$ and $a_5 = 42$, I was already congratulating myself on being so sharp as to conjecture that a_{2k+1} is always a Catalan number, $\frac{1}{n+1}\binom{2n}{n}$. However, $a_7 = 126$ quickly brought me back to earth, for the next Catalan number is 132. In any case, it is evident that the table consists essentially of just the two sequences

$$1,2,3,6,9,18,27,\ldots,3^k,\ 2\cdot 3^k,\ldots,(k \geq 0),$$

and

$$1,2,4,6,12,18,36,\ldots,\ 4\cdot 3^{k-1}, 2\cdot 3^k,\ldots,(k \geq 1).$$

Now let us proceed with the following kingsize induction. For our induction hypothesis, let us assume that all ten of the entries given in columns $2k+1$ and $2k+2$ of the table are valid for some integer $k \geq 1$. Then it is a simple matter to verify from the recurrences that the entries in columns $2k+3$ and $2k+4$ are as shown in the table. Since these have the *same form* as the corresponding entries in columns $2k+1$ and $2k+2$, and since they are born out by columns 3 and 4, we conclude by induction that these formulas are valid for all $k \geq 1$. Thus, by simply adding the entries in the columns, we obtain $a_1 = 5$, and, for $k \geq 0$,

$$a_{2k+2} = 8\cdot 3^k$$
$$a_{2k+3} = 8\cdot 3^k + 2\cdot 3^{k+1}.$$

Since these cases resemble each other so closely, we can roll them together to get the single formula

$$a_n = 8\cdot 3^{[(n-2)/2]} + (1-(-1)^n)\cdot 3^{[n/2]},$$

where the square brackets indicate "integer part."

8. From Australia

(*Crux Mathematicorum*, 1987, 276)

O_1, O_2, and O_3 are the centers of three circles K_1, K_2, and K_3 which pass through a common point P (Figure 19). Their second points of intersection are A, B, and C, as shown. From an arbitrary point X on K_1, XA is extended to meet K_2 at Y, and XC to meet K_3 at Z. For all choices of X on K_1, prove that

(i) Y, B, and Z are collinear, and that
(ii) the area of ΔXYZ is never more than four times the triangle of centers $O_1O_2O_3$.

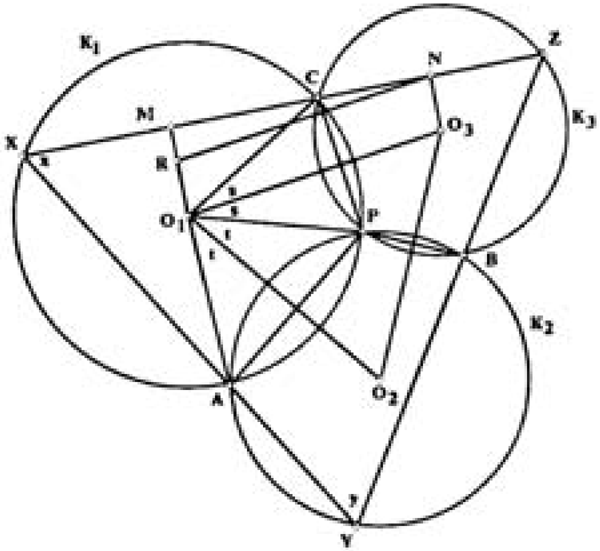


FIGURE 19

Solution

(i) Choosing X on K_1 determines the position of Y on K_2, and with both X and Y in place, the lines XC and YB are set. Now, XC crosses K_3 at Z, and we are required to show that YB also goes through this point Z. In other words, we would like to show that *XC and YB meet on the circle K_3*.

Now, if the angles at X and Y are x and y, as shown, the opposite angles, CPA in K_1 and APB in K_2, are their supplements $180° - x$ and $180° - y$. Thus the angle CPB in K_3 is

$$360° - (180° - x) - (180° - y) = x + y,$$

implying that the angle in the segment of K_3 that is on the *other* side of chord BC is $180° - (x + y)$. But, since the angles at X and Y are x and y, wherever the lines XC and YB might meet, they will form a triangle which has the angle $180° - (x + y)$ at the third vertex; that is to say, the chord CB will subtend an angle of $180° - (x + y)$ at their point of intersection. Being the very angle in the segment of K_3 which is cut off by the chord BC, we conclude that XC and YB must come together on K_3, as desired.

(ii) Since the line of centers O_1O_3 is the perpendicular bisector of the common chord CP of K_1 and K_3, C is the mirror image of P in O_1O_3. Hence

$$\measuredangle PO_1O_3 = \measuredangle CO_1O_3.$$

Similarly, P reflects into A in O_1O_2, and we have

$$\measuredangle PO_1O_2 = \measuredangle AO_1O_2.$$

Therefore

$$\measuredangle AO_1C = 2\cdot\measuredangle O_2O_1O_3.$$

But, in K_1, the angle AO_1C on chord AC is twice the angle x that AC subtends at X on the circumference. Hence, in $\measuredangle O_1O_2O_3$,

$$\measuredangle O_1 = x.$$

Similarly,

$$\measuredangle O_2 = y,$$

and the triangles XYZ and $O_1O_2O_3$ are equiangular and therefore similar.

Now, let perpendiculars O_1M and O_3N be drawn to XZ and let NR be parallel to O_1O_3, as shown. Then M and N are the midpoints of chords XC and CZ, making MN half as long as XZ. Also, in parallelogram RO_1O_3N, we have $O_1O_3 = RN$, the hypotenuse of right triangle RMN. Thus

$$O_1O_3 = RN \geq MN = \tfrac{1}{2}XZ,$$

and we have

$$\frac{XZ}{O_1O_3} \leq 2$$

(when XZ is parallel to O_1O_3, the triangle RMN collapses, making $RN = MN$ and $XZ = 2\cdot O_1O_3$).

Thus the ratio of corresponding sides in these similar triangles does not exceed 2, and since the areas of similar triangles are proportional to the *squares* on corresponding sides, we have the desired.

$$\frac{\Delta XYZ}{\Delta O_1O_2O_3} \leq 2^2 = 4.$$

•

9. From Finland

(*Crux Mathematicorum*, 1987, 309; a similar solution is given in 1989, 165).

Let $A = \{a_1 < a_2 < a_3 < \cdots\}$ be an infinite increasing sequence of positive integers in which the number of prime factors of each term, counting repeated factors, is never more than 1987. Prove that it is always possible to extract from

A an infinite subsequence

$$B = \{b_1 < b_2 < b_3 < \cdots\}$$

such that the greatest common divisor (b_i, b_j) is the *same number* for every pair of its terms.

Solution

Since the number of primes is infinite, one can never run out of factors with which to construct the terms of A. If the number of primes used collectively in all the a_i were only some *finite* number $n - 1$, then there would be only n choices for each of the 1987 factors in the composition of each a_i, each factor being either one of the $n - 1$ primes or just the number 1, and it would be possible to construct only n^{1987} different terms a_i. Thus an infinite sequence A requires that an *infinity* of different primes be employed in its construction.

Now, although no prime can occur more often than 1987 times in any particular term, each prime in A may be used throughout the sequence as often as desired. Clearly there are two cases:

either (i) no prime p is used infinitely often in the sequence,
or (ii) at least one prime p occurs infinitely often among the factors of the a_i.

Let us consider these possibilities in turn.

Case (i) No prime p Occurs Infinitely Often: In this case, no matter how often the 1987 or fewer prime factors of a_1 might occur throughout the rest of the sequence, there comes a term a_i which contains the last of them. Thus every term $a_j > a_i$ is relatively prime to a_1. Accordingly, let the desired subsequence B begin with $b_1 = a_1$ and $b_2 =$ any such a_j.

Similarly, all the primes in b_2 peter out at some term a_k, beyond which all the terms a_t are relatively prime to *both* b_1 and b_2. Let $b_3 =$ any such a_t. Thus, continuing to choose b_n at a point beyond which all the prime factors of b_{n-1} have petered out, each new term b_n lies beyond the cut-off points of all prior terms $b_1, b_2, \ldots, b_{n-1}$, and is therefore relatively prime to each of them. It follows that every pair (b_i, b_j) must be relatively prime, since one of b_i, b_j must precede the other in the sequence, and an acceptable subsequence B is thus generated.

Case (ii) At Least One Prime p Occurs Infinitely Often: Any prime p_1 that occurs infinitely often must occur in an infinity of the a_i, since it can't occur more than 1987 times in any one of them. Now, in the prime decompositions of the terms in which p_1 occurs, the powers p_1^t carry exponents in the

finite range $\{1, 2, \ldots, 1987\}$, and so some exponent r_1 must occur infinitely often; that is to say, there is an infinity of terms in which p_1 occurs to exactly the same power r_1. Let us discard all the other terms of A and consider just the subsequence A_1 of terms whose prime decompositions all contain $p_1^{r_1}$.

Obviously, then, in each term of A_1, the prime p_1 occupies r_1 of the 1987 places for factors. Now, in the overall collection of the *other* prime factors in A_1, some prime p_2 might also occur infinitely often. If so, then, just as for p_1, there must be an infinite subsequence A_2 of A_1 in which p_2 occurs to exactly the same power $p_2^{r_2}$. In this case, let us again reduce our considerations to A_2. Again, among the remaining prime factors of A_2, some prime p_3 might occur infinitely often and lead us to another subsequence A_3 in which every term contains the entire product $p_1^{r_1} p_2^{r_2} p_3^{r_3}$. However, there is a limit to the number of such primes p_i, for there are only 1987 places for prime factors in any term a_j. The composite factor $p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ occupies $r_1 + r_2 + \cdots + r_t$ of the places in *every* term of the resulting subsequence A_t. This must stop short of all 1987 places or else all the terms of A_t would be the same. Thus there comes a time when no prime, besides the already acknowledged $p_1, p_2, \ldots, p_t$, occurs infinitely often among the terms of A_t. If at this point we let the common factor $p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t} = P$, then this final A_t may be written in the form

$$A_t = \{Pc_1 < Pc_2 < Pc_3 < \cdots\}.$$

In this case, the sequence of auxiliary factors

$$C = \{c_1 < c_2 < c_3 < \cdots\}$$

is a sequence in which each term is a product of not more than

$$1987 - (r_1 + r_2 + \cdots + r_t)$$

factors and in which *no prime factor occurs infinitely often*. By case (i) above, then, C contains an infinite subsequence

$$\{c_1 < c_j < c_k < \cdots\}$$

in which every pair of terms is relatively prime. Finally, then, the corresponding subsequence B of A_t, given by

$$B = \{Pc_1 < Pc_j < Pc_k < \cdots\},$$

has the property that the greatest common divisor of every pair of its terms (Pc_s, Pc_t) is constantly equal to P.

Two Problems from the 1988 USA Olympiad

(*Crux Mathematicorum*, 1988, 164)

Problem 1

Let S be the set $\{1, 2, \ldots, 20\}$, and let each 9-element subset of S be assigned a label from S itself. Thus, for example,

$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ might be labelled 4,

$\{1, 2, 3, 4, 5, 6, 7, 8, 10\}$ might also be labelled 4, and

$\{1, 2, 3, 4, 5, 6, 7, 8, 11\}$ might be labelled 17, and so on.

There are $\binom{20}{9} = 167960$ different 9-element subsets of S, and so, on the average, each of the 20 labels gets used 8398 times.

Prove that no matter how these labels might be assigned, there always exists a 10-element subset T of S with the property that

no element of T is the label for the subset determined by the other 9 elements of T.

Solutions

In trying to solve this problem, I found myself waffling between the direct and indirect approaches; failures to construct T directly by substituting for incompatible members alternated with failures to find a contradiction to the denial of T. On its own, the following short solution gives the impression that the problem is extremely simple. But that is a very misleading impression, for I was a long time attaining this enlightened point of view.

Proceeding indirectly after all, suppose that T does not exist. In this case, every 10-element subset A contains at least one element x which labels the subset X consisting of the other 9 elements of A. Accordingly, let a set M be compiled in which all $\binom{20}{10}$ 10-element subsets of S are written in the form $(X; x)$, where x is the label assigned to the 9-element complementary subset X:

$$M = \{(X; x), (Y; y), (Z; z), \ldots\}.$$

Now, there are only $\binom{20}{9}$ 9-element subsets of S, and because the binomial coeffcients $\binom{20}{r}$ increase up to the middle one, namely $\binom{20}{10}$, there are more 10-element subsets in M than there are 9-element subsets to go around. Therefore some two subsets A and B in M must display the same 9-element subset X:

$$A = (X; x), \quad B = (X; y).$$

Since A and B are different, then x and y must be different labels assigned to the same 9-element subset X, and we have already reached a contradiction to complete the solution.

Problem 2

Let segments be drawn from the incenter I to the vertices of $\triangle ABC$ to partition the triangle into three smaller triangles. If O_1, O_2, O_3 are the circumcenters of these little triangles, prove that the circumcircles of triangles $O_1O_2O_3$ and ABC are concentric, that is, that the circumcenter O of $\triangle ABC$ is also the circumcenter of $\triangle O_1O_2O_3$.

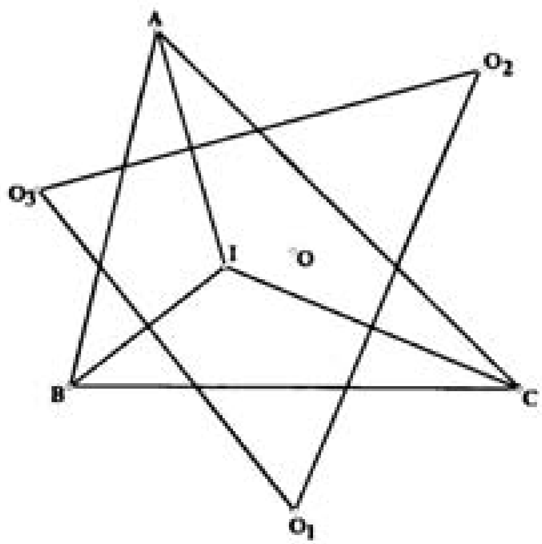


FIGURE 20

Solution

Since I is the incenter of $\triangle ABC$, the segments from I to the vertices bisect the angles of the triangle. In $\triangle IAB$, then, the angles at A and B add up to $\frac{1}{2}(A+B) < \frac{1}{2}(A+B+C) = 90°$, implying that the angle at I is obtuse. Now, the circumcenter of an obtuse triangle lies outside the triangle, and so O_3 is outside $\triangle ABC$ as shown. Similarly for O_1 and O_2. This observation is not crucial to the argument, but it does make things slightly simpler.

Because the circumcenter of a triangle lies on the perpendicular bisector of each side, both O and O_1 lie on the perpendicular bisector of BC, implying that OO_1 actually is the perpendicular bisector of BC. Similarly OO_3 and O_1O_3 are the perpendicular bisectors, respectively, of AB and IB. Labelling points of intersection as shown in Figure 21, the equal vertically opposite angles at D imply the angles at B and O_1 in right-triangles DO_1E and DBF are equal, and we have

$$\measuredangle O_1 = \tfrac{1}{2}\measuredangle B.$$

Similarly,

$$\measuredangle O_3 = \text{the other half of } \measuredangle B,$$

and we conclude that

$$\measuredangle O_1 = \measuredangle O_3,$$

making OO_1 and OO_3 the equal arms in isosceles triangle OO_1O_3.

Similarly, $OO_1 = OO_2$, making $OO_1 = OO_2 = OO_3$, and the conclusion follows.

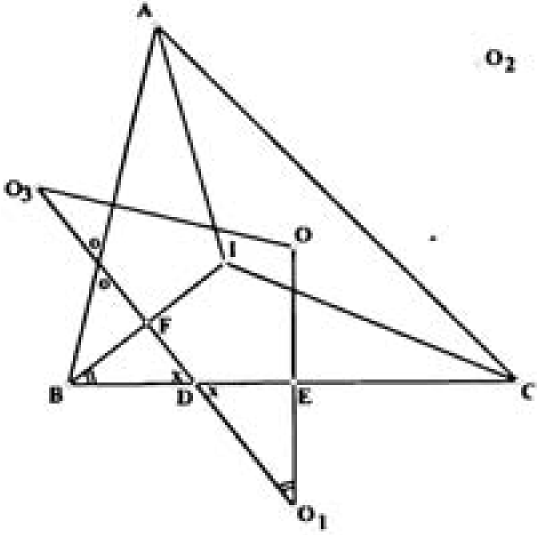


FIGURE 21

From the 1988 International Olympiad

(*Crux Mathematicorum*, 1988, 197)

Show that the set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

Solution

It's remarkable how they generally come up with a good problem involving the number of the year of the contest; I find it especially engaging in this problem that simple numbers like 70 and $\frac{5}{4}$ lead to a sum of intervals that is exactly 1988. I have no idea where they get these problems, but this one is a lovely gem.

After all kinds of fruitless investigations, I finally gave more serious attention to charting the behavior of the function $S(x) = \sum_{k=1}^{70} \dfrac{k}{x-k}$ with a graph, which I expect many of the contestants did in the first place. Clearly the function is discontinuous at $x = k$ for $k = 1, 2, \ldots, 70$, but is continuous in the open intervals between these integers. Also, the function goes to $-\infty$ as x approaches k from below and to $+\infty$ as x approaches k from above. Thus the graph crosses the line $y = \frac{5}{4}$ in each of the intervals $(k, k+1)$ for $k = 1, 2, \ldots, 69$. For $x > 70$, the biggest term in $S(x)$, namely $\dfrac{70}{x-70}$, gets arbitrarily small as x increases, showing that the positive x-axis is an asymptote. Therefore from a

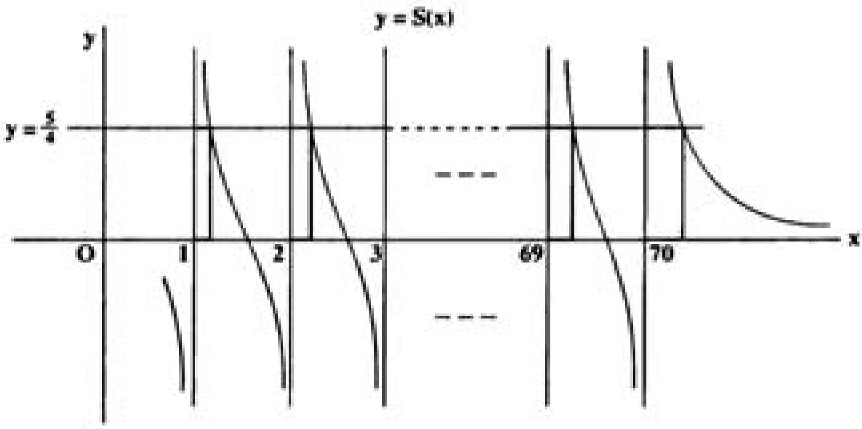


FIGURE 22

sketch of the graph of $y = S(x)$ we can see that the set of values of x for which $S(x) \geq \frac{5}{4}$ consists of 70 half-open intervals—open on the left and closed on the right—which begin respectively at the integer points $x = 1, 2, \ldots, 70$.

In fact, in the graph of $y = S(x) - \frac{5}{4}$, these intervals occurs on the x-axis as the intervals between k and the root x_k of the equation $S(x) - \frac{5}{4} = 0$ which lies between k and the $k + 1$. The length of the kth interval, then, is simply $x_k - k$, and the sum of all 70 intervals is

$$\begin{aligned}&(x_1 - 1) + (x_2 - 2) + \cdots + (x_{70} - 70)\\ &= (x_1 + x_2 + \cdots + x_{70}) - (1 + 2 + \cdots + 70).\end{aligned}$$

It remains, then, only to show that the sum of the roots of $S(x) - \frac{5}{4} = 0$ is $1988 + (1 + 2 + \cdots + 70)$. Thankfully, this is a straightforward calculation for, if $S(x) - \frac{5}{4} = ax^{70} + bx^{69} + \cdots$, the sum of the roots is simply $-\frac{b}{a}$.

Clearing of fractions in

$$S(x) - \frac{5}{4} = \frac{1}{x-1} + \frac{2}{x-2} + \cdots + \frac{70}{x-70} - \frac{5}{4} = 0,$$

we get

$$\begin{aligned}&4(x-2)(x-3)\cdots(x-70) + 4\cdot 2(x-1)(x-3)\cdots(x-70)\\ &\quad + \cdots + 4\cdot 70(x-1)(x-2)\cdots(x-69) - 5(x-1)(x-2)\cdots(x-70) = 0,\\ &\quad -5x^{70} + x^{69}[4\cdot 1 + 4\cdot 2 + \cdots + 4.70 - 5(-1-2-\cdots-70)] + \cdots = 0,\\ &\quad -5x^{70} + x^{69}[4(1+2+\cdots+70) + 5(1+2+\cdots+70)] + \cdots = 0,\\ &\quad -5^{70} + 9(1+2+\cdots+70)x^{69} + \cdots = 0,\end{aligned}$$

and the sum of the roots is

$$\begin{aligned}\frac{9(1+2+\cdots+70)}{5} &= (1+2+\cdots+70)+\frac{4}{5}(1+2+\cdots+70)\\ &= (1+2+\cdots+70)+\frac{4}{5}\cdot\frac{70\cdot 71}{2}\\ &= (1+2+\cdots+70)+28\cdot 71\\ &= (1+2+\cdots+70)+1988, \text{ as desired.}\end{aligned}$$

A Geometric Gem of Duane DeTemple

The beautiful result of this section is a discovery of Duane DeTemple of Washington State University.

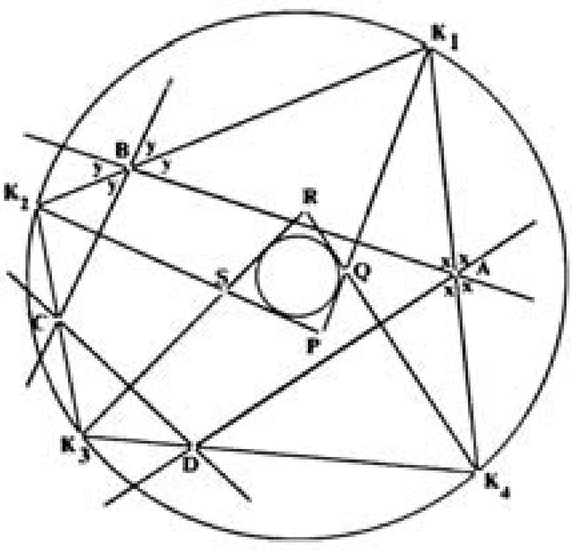


FIGURE 23

Let the sides of convex quadrilateral $ABCD$ be extended and the bisectors of the exterior angles meet at points K_1, K_2, K_3, and K_4, as shown. Since a point on the bisector of an angle is equidistant from the arms of the angle, K_1 is equidistant from all three of the lines AB, DA, and CB; thus K_1 is the center of the

excircle which touches AB internally, and altogether K_1, K_2, K_3, K_4 are the four excenters of $ABCD$. Prove that $K_1K_2K_3K_4$ is always a *cyclic* quadrilateral.

Now suppose a perpendicular is drawn from each K_i to its associated side of $ABCD$, that is, from K_1 to AB, from K_2 to BC, etc., to form another quadrilateral $PQRS$, as shown. Prove the remarkable fact that $PQRS$ always has an incircle and that this incircle is always concentric with the circle around $K_1K_2K_3K_4$.

Solution

It is not difficult to show that $K_1K_2K_3K_4$ is always cyclic. Referring to Figure 24, each angle marked x is one-half the exterior angle at A, that is,

$$x = \tfrac{1}{2}(180° - A);$$

similarly,

$$y = \tfrac{1}{2}(180° - B).$$

Hence the angle z at K_1 is simply

$$z = 180° - x - y = \tfrac{1}{2}(A + B).$$

Similarly, the angle w at K_3 is

$$w = \tfrac{1}{2}(C + D),$$

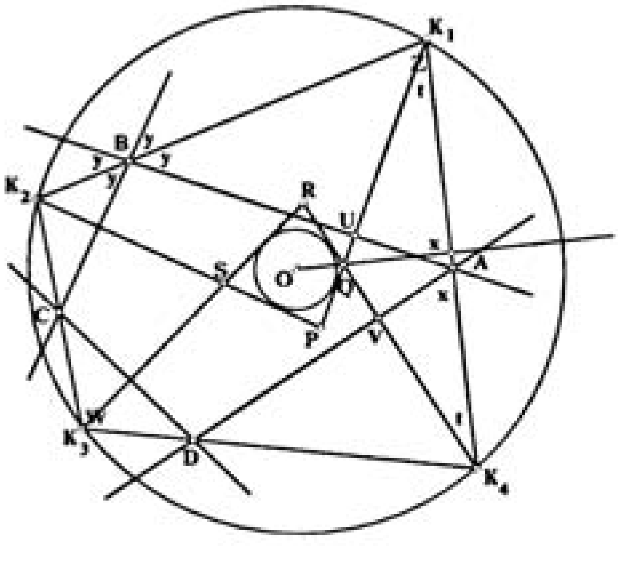


FIGURE 24

and therefore $K_1K_2K_3K_4$ is cyclic because

$$z + w = \tfrac{1}{2}(A + B + C + D) = \tfrac{1}{2}\cdot 360^\circ = 180^\circ.$$

Now, in right-triangles K_1UA and K_4AV, the angles x at A are equal, implying that the angles t at K_1 and K_4 are also equal. This makes triangle K_1K_4Q isosceles and it follows that the bisector of $\measuredangle Q$ in this triangle is the perpendicular bisector of the base K_1K_4. But K_1K_4 is a chord of the circle around $K_1K_2K_3K_4$, and hence this perpendicular bisector goes through the center O of this circle; that is to say, in quadrilateral $PQRS$, OQ bisects $\measuredangle Q$. Similarly, OP, OR, and OS are the bisectors of the other angles in $PQRS$, and it follows that O is equidistant from all four of its sides!

Professor DeTemple observes that there are occasions when $PQRS$ degenerates to a point, in which case, as you might expect, the point is simply the center O of the circle about $K_1K_2K_3K_4$. He also observes that the shape of $K_1K_2K_3K_4$ has greater *order* than $ABCD$:

(i) whatever the shape of $ABCD$, $K_1K_2K_3K_4$ is *cyclic*;
(ii) when $ABCD$ is a parallelogram, $K_1K_2K_3K_4$ is not only a parallelogram but a *rectangle*;
(iii) when $ABCD$ is a rectangle, $K_1K_2K_3K_4$ is not only a rectangle but a *square*.

A Kiev Olympiad Problem

I would not like to assume that grade 9 in Kiev corresponds to the first year of high school in North America. In any case, the following engaging problem comes from the 1954 Kiev olympiad for grade 9 students.

> A circle is inscribed in a triangle and a square is circumscribed about the circle. Prove that
>
> *more than half the perimeter* of the square
> *lies inside or on* the triangle.

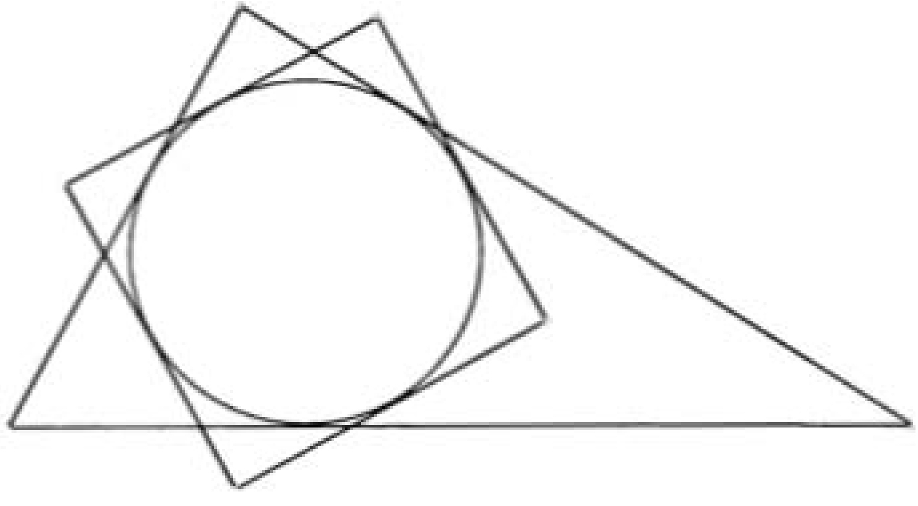


FIGURE 25

Solution

In general, three corners of the square stick out of the triangle. Let the legs of the right-triangles thus cut off from the square have lengths p, q, r, s, t, u, and let the pairs of equal tangents to the circle have lengths a, b, c, d, e, f, and g, as shown.

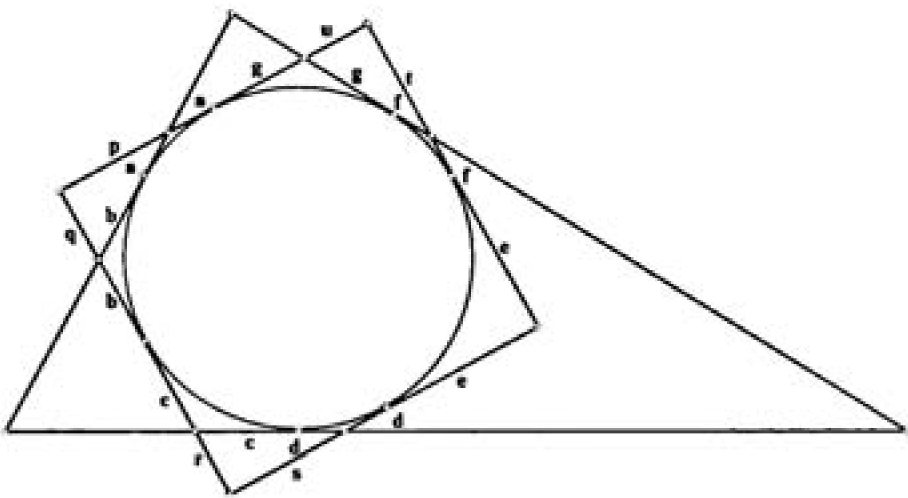


FIGURE 26

Then the length of perimeter that lies inside the triangle is

$$(a + b) + (c + d) + (f + g) + 2e,$$

where e is the radius of the circle, and the perimeter that is outside the triangle is

$$(p + q) + (r + s) + (t + u).$$

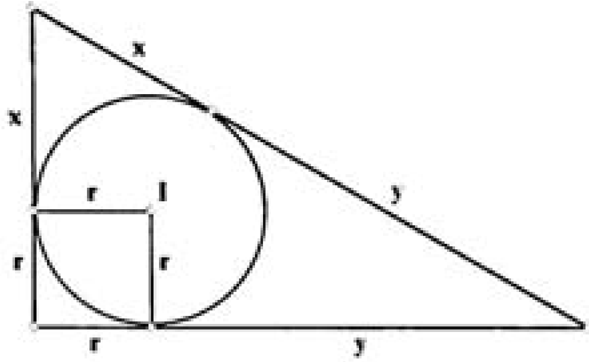
Now, in a right-angled triangle, we have the special relation that

(the sum of the legs) − (the hypotenuse) = the diameter of the incircle.

This is clear from Figure 27.

Consequently we have

$$(p + q) - (a + b) = d_1$$
$$(r + s) - (c + d) = d_2$$
$$(t + u) - (f + g) = d_3$$



$$[(x + r) + (r + y)] - (x + y) = 2r$$

FIGURE 27

where d_1, d_2, d_3 are the respective diameters of the incircles of the overlapping right-triangles. Thus

$$(p+q)+(r+s)+(t+u)-[(a+b)+(c+d)+(f+g)]$$
$$= d_1 + d_2 + d_3.$$

Subtracting $2e$ from each side gives

$$(p+q)+(r+s)+(t+u)-[(a+b)+(c+d)+(f+g)+2e]$$
$$= d_1 + d_2 + d_3 - 2e,$$

i.e.,

$$\text{outer perimeter} - \text{inner perimeter} = d_1 + d_2 + d_3 - 2e,$$

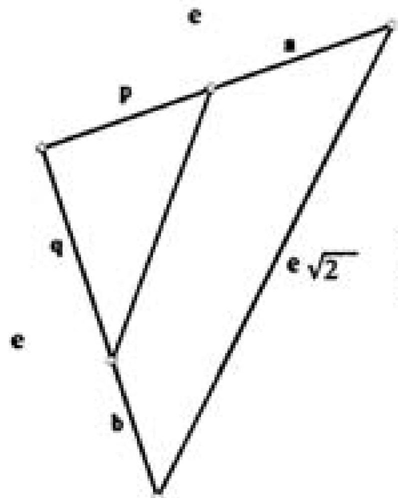
and multiplying through by -1, we get

$$\text{inner perimeter} - \text{outer perimeter} = 2e - (d_1 + d_2 + d_3).$$

It remains only to show that this difference is positive, or, equivalently, that

$$2e > d_1 + d_2 + d_3.$$

There is no denying that it looks as if relevent information about these diameters might be hard to come by—until you notice that each of these little triangles is embedded in an isosceles right-angled triangle having legs equal to the radius e of the circle. For example, consider the right triangle with legs p and q (Figure 28).



$$p + a = e,$$
$$q + b = e.$$

FIGURE 28

Clearly, then, each of the diameters d_1, d_2, d_3 is not greater than the diameter d of the incircle of such an isosceles right-triangle. But d is just

$$\begin{aligned} d &= (e+e) - e\sqrt{2} \\ &= e(2-\sqrt{2}) \\ &= e(2 - 1 \cdot 4142\ldots) \\ &< e(\cdot 6) \\ &= \frac{3}{5}e. \end{aligned}$$

Hence

$$d_1 + d_2 + d_3 < \frac{9}{5}e < 2e, \quad \text{as desired.}$$

Some Student Favorites

The problems in this section are taken from a miscellaneous collection entitled Forty Exciting Problems that was put together by two students—Waterloo's Frank D'Ippolito and Toronto's Ravi Vakil. At the end, a few other problems from this collection are given as exercises.

Problem 1

Each of the numbers $x_1, x_2, \ldots, x_n$ is either $+1$ or -1. If the sum

$$S = x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + \cdots + x_nx_1x_2x_3 = 0,$$

prove that n must be a multiple of 4.

Solution

Since each x_i is either $+1$ or -1, each term $x_ax_{a+1}x_{a+2}x_{a+3}$ is also either $+1$ or -1, and since the sum of the n such terms is zero, there must be the same number of terms equal to $+1$ as equal to -1, namely $n/2$ of each kind.

Suppose that p of the x_i are $+1$'s. Since each x_i occurs in four of the terms $x_ax_{a+1}x_{a+2}x_{a+3}$, there will be a total of $4p$ occurrences of a factor x_i equal to $+1$ throughout the sum S. Each term that is equal to $+1$ will contain an *even* number of factors $x_i = +1$, and each term equal to -1 will contain an *odd* number of them. Altogether, then, the total number of occurrences of a factor equal to $+1$ is

$$\begin{aligned} 4p = {} & \text{(a sum of } n/2 \text{ even integers)} \\ & + \text{(a sum of } n/2 \text{ odd integers)}. \end{aligned}$$

Since any sum of even integers is even, and the total $4p$ is even, it follows that the sum of the $n/2$ odd integers must also be even. But the sum of a collection of odd integers is even only when there is an *even number* of them. Hence $n/2$ must be even, making n a multiple of 4.

Problem 2

Let $f(x) = x^n$, where n is a *fixed* positive integer and x runs through all the positive integers $1, 2, 3, \ldots$. The digits of $f(1), f(2), \ldots$ are placed end-to-end to form an infinite decimal y_n:

$$y_n = 0.\left(\underset{f(1)}{\cdots\cdots}\right)\left(\underset{f(2)}{\cdots\cdots}\right)\left(\underset{f(3)}{\cdots\cdots}\right)\cdots.$$

Thus, for example,

$$y_2 = 0.149162536496481\ldots$$

and

$$y_3 = 0.182764125216343\ldots.$$

Is y_n ever a *rational* number for any value of n?

Solution

The answer is "No, y_n is always irrational."

The powers of 10 force every y_n to contain arbitrarily long strings of consecutive 0's:

$$f(10^k) = 10^{kn} = 10000\cdots 0.$$

But no rational decimal contains arbitrarily long strings of consecutive 0's except a terminating decimal. Thus, in order to be rational, y_n would have to terminate. But every contribution x^n begins with a nonzero digit, showing that y_n does not terminate.

Problem 3

Let $a_1 < a_2 < \cdots < a_{43} < a_{44}$ be positive integers not exceeding 125. Prove that among the 43 consecutive differences $d_i = a_{i+1} - a_i$, some value must occur at least 10 times.

Solution

Beginning with a_1, any of the a_i can be obtained by adding the differences $d_1, d_2, \ldots, d_{i-1}$:

$$a_i = a_1 + d_1 + d_2 + \cdots + d_{i-1}.$$

Hence

$$a_{44} = a_1 + d_1 + d_2 + \cdots + d_{43}.$$

Since the a's are all different, each $d_i \geq 1$. Now, if no value were to occur as often as 10 times among the d_i, then, in particular, $d_i = 1$ could not occur more than 9 times; nor could $d_i = 2$, or 3, or 4. Thus the 36 smallest d_i must amount to at least $9(1 + 2 + 3 + 4) = 90$. The remaining seven d_i must each be at least 5, and we have that

$$a_{44} \geq a_1 + 9(1 + 2 + 3 + 4) + 7(5) = a_1 + 125 > 125.$$

Thus a_{44} exceeds its limit of 125 unless some d_i occurs at least 10 times.

Problem 4

Two circles are internally tangent at a point T. A chord AB of the outer circle touches the inner circle at the point P. Prove that TP always bisects $\measuredangle ATB$.

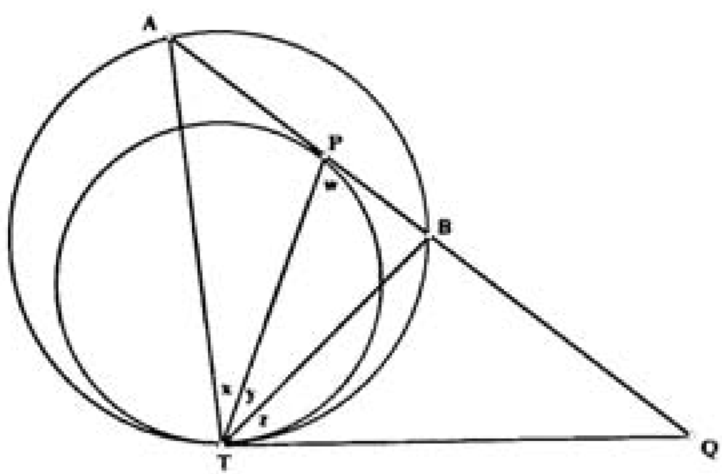


FIGURE 29

Solution

Let AB extended meet the common tangent at Q, and let the angles x, y, z, and w be marked as in Figure 29.

Then QT and QP are equal tangents to the inner circle, making triangle TQP isosceles and

$$w = y + z.$$

With regard to the outer circle, TQ is a tangent and TB is a chord; thus the angle z between them is equal to the angle in the segment on the other side of the chord, that is,

$$z = \measuredangle A.$$

Now, the exterior angle w of triangle APT is equal to the sum of the interior angles at the other two vertices, and we have

$$w = x + \measuredangle A = x + z.$$

Hence

$$w = y + z = x + z, \text{ giving } y = x, \text{ as required.}$$

Problem 5

Consider the trio of integers (2,5,13). The number which is one less than the product of two of them is in each case a perfect square:

$$2(5) - 1 = 9, \quad 2(13) - 1 = 25, \quad 5(13) - 1 = 64.$$

Prove, however, that this is never the case if a new positive integer d is appended to the trio, i.e., that one less than the product of some two integers in the quadruple $\{2, 5, 13, d\}$ will fail to be a square for every positive integer d.

Solution

Since $ab - 1$ *is* a perfect square for all three choices of a and b from the original trio $\{2, 5, 13\}$, we need only show that, for every positive integer d, one of the numbers

$$x = 2d - 1, \quad y = 5d - 1, \quad z = 13d - 1$$

must fail to be a perfect square.

Let us consider the residue class of d modulo 4. We observe that, modulo 4, a square is always congruent either to 0 or 1; $(2a)^2 = 4a^2$, and $(2a + 1)^2 =$

$4(a^2 + a) + 1$, and that therefore the sum of two squares, $a^2 + b^2$, is never congruent to 3 (mod 4).

(i) $d \equiv 0$ or 2: In this case, $x = 2d - 1 \equiv 3$, and is thus not a square.
(ii) $d \equiv 3$: Similarly, this makes $y = 5d - 1 \equiv 2$ and not a square.

(iii) $d \equiv 1$: In this case, $d = 4k + 1$ for some integer $k \geq 0$, making $x = 8k + 1$,
$y = 20k + 4 = 4(5k + 1)$, $z = 52k + 12 = 4(13k + 3)$.

Now let us suppose that all three of x, y, and z are perfect squares and try to deduce a contradiction.

If $y = 4(5k + 1)$ is a perfect square, then, because its factor 4 is a square, the complementary factor $5k + 1$ must also be a square. Similarly for the factor $13k + 3$ of z, and each of $8k + 1, 5k + 1$, and $13k + 3$ must be a square. Accordingly, letting

$$8k + 1 = a^2, \quad 5k + 1 = b^2, \quad 13k + 3 = c^2,$$

we observe that

$$a^2 + b^2 = c^2 - 1.$$

Now, as noted above, $c^2 \equiv 0$ or 1 (mod 4), and we have that

$$a^2 + b^2 = c^2 - 1 \equiv 3 \text{ or } 0 \pmod 4.$$

Since $a^2 + b^2$ is never congruent to 3 (mod 4), it follows that

$$a^2 + b^2 \equiv 0 \pmod 4.$$

Since each of a^2 and b^2 is congruent to 0 or 1 (mod 4), this result can only be achieved if

$$a^2 \equiv b^2 \equiv 0 \pmod 4.$$

That is to say, a^2 must be an *even* number. However, this is clearly not so, since $a^2 = 8k + 1$.

Problem 6

Prove that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$$

for all positive integers $n \geq 2$.

Solution

The n roots of $x^n = 1$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

Now,

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

Since the factor $x - 1$ accounts for the root $x = 1$, the complementary factor must give rise to all the other roots, and we have the factoring

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = (x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}).$$

Putting $x = 1$ in this identity yields

$$n = (1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1}),$$

and, taking absolute values, we obtain

$$n = |1-\omega|\,|1-\omega^2|\cdots|1-\omega^{n-1}| = \prod_{k=1}^{n-1}|1-\omega^k|.$$

It remains only to show that $|1-\omega^k| = 2\sin\frac{\pi k}{n}$.

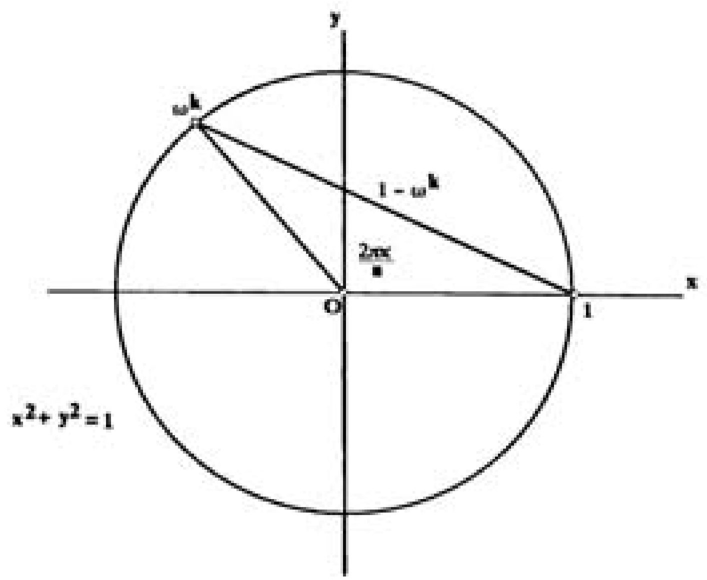


FIGURE 30

The triangle of vectors in the complex plane that is determined by the vectors $1, \omega^k$, and $1 - \omega^k$ is shown in Figure 30. By the law of cosines, the magnitude of $1 - \omega^k$ is given by

$$|1 - \omega^k|^2 = 1^2 + 1^2 - 2\cos\frac{2\pi k}{n}$$
$$= 2\left[1 - \left(1 - 2\sin^2\frac{\pi k}{n}\right)\right] = 4\sin^2\frac{\pi k}{n}.$$

Thus

$$|1 - \omega^k| = 2\sin\frac{\pi k}{n}, \text{ as desired,}$$

and the required relation follows directly.

Problem 7

$(a_1, a_2, \ldots, a_n)$ is a permutation of $\{1, 2, 3, \ldots, n\}$. What is the *average value* of the sum

$$S = (a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_{n-1} - a_n)^2$$

taken over all $n!$ permutations?

Solution

Let (i, j) be any ordered pair from $\{1, 2, \ldots, n\}$. Then there are $(n-2)!$ permutations that set $a_1 = i$ and $a_2 = j$, giving $(n-2)!$ times when the first term of S is

$$(a_1 - a_2)^2 = (i - j)^2,$$

for a contribution of $(n-2)!(i-j)^2$ to the sum $\sum S$ of all $n!$ values of S. The same pair (i, j) causes each of the $n - 1$ terms $(a_k - a_{k+1})^2$ to make an equal contribution to $\sum S$, giving a total of

$$(n-1)(n-2)!(i-j)^2 = (n-1)!(i-j)^2 \quad \text{for the pair } (i, j).$$

Adding over all ordered pairs, the grand total $\sum S$ is given by

$$\sum S = \sum_{(i,j)} (n-1)!(i-j)^2,$$

for each term in every S belongs to some ordered pair (i, j).

Hence the required average is

$$\begin{aligned}
A &= \frac{1}{n!}\sum S = \frac{1}{n}\sum_{(i,j)}(i-j)^2. \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(i-j)^2 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(i^2-2ij+j^2) \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}ni^2 + \sum_{j=1}^{n}nj^2 - 2\left(\sum_{i=1}^{n}i\right)\left(\sum_{j=1}^{n}j\right)\right] \\
&= \frac{1}{n}\left[2n\cdot\frac{n(n+1)(2n+1)}{6} - 2\left[\frac{n(n+1)}{2}\right]^2\right] \\
&= \frac{n(n+1)}{6}[2(2n+1)-3(n+1)] \\
&= \frac{n(n+1)(n-1)}{6} \\
&= \binom{n+1}{3}.
\end{aligned}$$

This neat combinatorial answer makes one wonder whether there is some ingenious combinatorial argument that gives the result quickly and easily.

Exercises

1. If $a+b+c=0$, prove the engaging relation

$$\frac{a^5+b^5+c^5}{5} = \left(\frac{a^3+b^3+c^3}{3}\right)\left(\frac{a^2+b^2+c^2}{2}\right).$$

2. Evaluate

$$\sum_{k=1}^{n}\frac{k}{k^4+k^2+1}.$$

3. (a) Find a polynomial with integer coefficients having $\sqrt{2}+\sqrt{3}$ as a root.
   (b) Find a polynomial with integer coefficients having $\sqrt{2}+\sqrt[3]{3}$ as a root.

4. If $x^3+y^3+z^3=0$ has a solution in integers, prove that one of x, y, z, must be a multiple of 7.

Four Unused Problems from the 1988 International Olympiad

1. From Bulgaria

(*Crux Mathematicorum*, 1988, 225)

Suppose the sequence $\{a_n\}$ is defined by

$$a_0 = 0, \quad a_1 = 1, \quad \text{and} \quad \text{for} \quad n \geq 2, \quad a_n = 2a_{n-1} + a_{n-2}.$$

Thus the sequence begins

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, \ldots,$$

As far as this small sample goes, then,

every second term, beginning at a_2, is divisible by 2, and these are the only terms that are divisible by 2;

every fourth term, beginning at a_4, is divisible by 4, and these are the only terms that are divisible by 4;

every eighth term, beginning at a_8, is divisible by 8, and these are the only terms that are divisible by 8.

Prove the general property that, for all positive integers k, a_n is divisible by 2^k if and only if n itself is divisible by 2^k.

Solution

I am not at all sure that the proposers expected the contestants to devise something equivalent to the following sophisticated approach but, in any

event, it is a very nice way to solve the problem. We take our cue from the treatment of the Fibonacci and Lucas sequences in chapter 8 of my *Mathematical Gems III* (vol. 9, Dolciani Mathematical Expositions, MAA, 1985).

The first step is to derive a formula for the general term a_n. This can be done with a standard application of generating functions, but we may also proceed very neatly as follows. If x is a root of the equation

$$x^2 = 2x + 1,$$

then, for all $n \geq 2$, we can easily prove by induction that

$$x^n = a_n x + a_{n-1}:$$

this is given to be true for $n = 2$, and if $x^{n-1} = a_{n-1}x + a_{n-2}$, then

$$\begin{aligned} x^n = x \cdot x^{n-1} &= a_{n-1}x^2 + a_{n-2}x \\ &= a_{n-1}(2x+1) + a_{n-2}x \\ &= (2a_{n-1} + a_{n-2})x + a_{n-1} \\ &= a_n x + a_{n-1}. \end{aligned}$$

Now, the roots of $x^2 = 2x + 1$, i.e., of $x^2 - 2x - 1 = 0$, are

$$\alpha = 1 + \sqrt{2} \quad \text{and} \quad \beta = 1 - \sqrt{2}.$$

Note that $\alpha\beta = -1$. Hence, for all $n \geq 2$,

$$\alpha^n = a_n\alpha + a_{n-1},$$

and

$$\beta^n = a_n\beta + a_{n-1}.$$

Subtracting gives

$$\alpha^n - \beta^n = a_n(\alpha - \beta),$$

and we have

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Clearly this also holds for $n = 0$ and $n = 1$.

Next, let us introduce a companion sequence $\{b_n\}$ defined by $b_0 = 2$, and for $n \geq 1$,

$$b_n = a_{n-1} + a_{n+1}.$$

Then

$$\{a_n\} = 0, 1, 2, 5, 12, 29, 70, \ldots,$$

and

$$\{b_n\} = 2, 2, 6, 14, 34, 82, \ldots .$$

Thus, from the formula for a_n, we obtain a formula for b_n:

$$\begin{aligned} b_n &= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta}[(\alpha^{n+1} + \alpha^{n-1}) - (\beta^{n+1} + \beta^{n-1})] \\ &= \frac{1}{\alpha - \beta}\left[\alpha^n\left(\alpha + \frac{1}{\alpha}\right) - \beta^n\left(\beta + \frac{1}{\beta}\right)\right]. \end{aligned}$$

Now,

$$\alpha + \frac{1}{\alpha} = \alpha + \frac{\beta}{\alpha\beta} = \alpha - \beta$$

(recall $\alpha\beta = -1$), and

$$\beta + \frac{1}{\beta} = \beta + \frac{\alpha}{\alpha\beta} = \beta - \alpha = -(\alpha - \beta).$$

Thus b_n is simply

$$b_n = \alpha^n + \beta^n.$$

From the factoring

$$a_{2n} = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\alpha - \beta}(\alpha^n + \beta^n),$$

we get the crucial result that

$$a_{2n} = a_n b_n.$$

It is easy to see that $\{b_n\}$ inherits from $\{a_n\}$ the recurrence relation

$$b_n = 2b_{n-1} + b_{n-2}:$$

by definition,

$$2b_{n-1} = 2a_{n-2} + 2a_n$$

and

$$b_{n-2} = a_{n-3} + a_{n-1},$$

giving

$$\begin{aligned} 2b_{n-1} + b_{n-2} &= (2a_{n-2} + a_{n-3}) + (2a_n + a_{n-1}) \\ &= a_{n-1} + a_{n+1} = b_n. \end{aligned}$$

From this it follows that b_{n-2} and b_n have the same parity, and since both b_1 and b_2 are even (even b_0 is even), it follows that b_n is always even. But we get another important property from this recursion, namely that b_n is always just *singly* even, that is, twice an odd number: suppose

$$b_{n-1} = 2c \quad \text{and} \quad b_{n-2} = 2d,$$

where c and d are odd, then

$$b_n = 2b_{n-1} + b_{n-2} = 4c + 2d = 2(2c + d),$$

which is also twice an odd number; the conclusion follows the observation that b_0 and b_1 are both $2 \cdot 1$.

Thus, in

$$a_{2n} = a_n b_n,$$

the factor b_n always provides exactly one factor 2.

Turning back to $\{a_n\}$, it follows in a similar way that a_n is even if and only if n is even: from $a_n = 2a_{n-1} + a_{n-2}$, a_n and a_{n-2} have the same parity, and since $a_0 = 0$ is even, so is a_{2n} for all n; since $a_1 = 1$ is odd, so is a_{2n+1} for all n. Thus the desired property "2^k divides a_n if and only if 2^k divides n" is valid for $k = 1$.

Proceeding by induction, suppose, for some $k \geq 2$, that

$$2^{k-1} \text{ divides } a_n \text{ if and only if } 2^{k-1} \text{ divides } n.$$

Now, 2^{k-1} can't possibly divide a_n unless a_n is even, and we have seen that this requires n to be even; and n is certainly even if it is divisible by 2^{k-1}. Thus n is always an even number in what follows, say $n = 2t$.

We have observed that, in

$$a_n = a_{2t} = a_t b_t,$$

b_t supplies exactly one factor 2. It follows, then, that

$$2^k \text{ divides } a_n \text{ if and only if } 2^{k-1} \text{ divides } a_t.$$

According to the induction hypothesis, however, 2^{k-1} divides a_t if and only if 2^{k-1} divides t itself, that is,

$$2^{k-1} \quad \text{divides} \quad \frac{n}{2},$$

or equivalently,

$$2^k \text{ divides } n.$$

Hence, by induction,

$$2^k \text{ divides } a_n \text{ if and only if } 2^k \text{ divides } n.$$

2. Proposed by East Germany

(*Crux Mathematicorum*, 1988, 257)

The lock on a safe consists of three wheels A, B, and C, each of which may be set in eight different positions. Due to a defect in the mechanism, the door will open when any two of the wheels are in the correct position. Thus anybody can open the safe in 64 tries (simply let B run through all 8 positions for *each* of the 8 settings for A). However, the safe can always be opened in far fewer tries than that. What is the minimum number of tries that can be guaranteed to open the safe?

Solution

(Due to my colleague Paul Schellenberg)

If the correct combination is (a, b, c), for wheels A, B, and C, respectively, then any trial which realizes any of the ordered pairs $(A, B) = (a, b)$, $(B, C) = (b, c)$, $(C, A) = (c, a)$ will open the safe. Thus we need to devise a scheme for testing the ordered pairs *three at a time*—each trial (A, B, C) tests a pair for each of (A, B), (B, C), (C, A)—which can be guaranteed to come across one of the key pairs in a minimum number of steps.

The first thing to observe is that, because the correct combination (a, b, c) contains *three* numbers, the pigeonhole principle implies that either

(i) some two of the numbers come from $(1, 2, 3, 4)$, or
(ii) some two of the numbers come from $(5, 6, 7, 8)$.

Let us concentrate temporarily, then, on testing the $4^2 = 16$ ordered pairs from $(1, 2, 3, 4)$:

$$(1, 1), (1, 2), (1, 3), \ldots, (4, 4).$$

Each of these ordered pairs needs testing in each of the three positions, for a total of $3 . 16 = 48$ tests. It is conceivable that this could be accomplished in just 16 trials if we could only manage to package them in threes (A, B, C) so that each ordered pair (x, y) occurs once as (A, B), once as (B, C), and once as (C, A). The ideal solution of this problem, then, would be a 3×16 array prescribing 16 trials in which each ordered pair occurs once in each pair of rows (A, B), (B, C), (C, A).

A	1		$\cdots$	–	$\cdots$	2
B	2		$\cdots$	1	$\cdots$	–
C	–		$\cdots$	2	$\cdots$	1

It turns out that this is quite easy to arrange as follows.

Since we want the numbers $(1,2,3,4)$ thoroughly mixed up, let's begin by constructing a 4×4 array (a so-called Latin square) in which each row and each column is one of the $4! = 24$ permutations of the numbers $(1,2,3,4)$. There are many ways to do this; for example, a simple shifting from row to row gives

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1. \end{array}$$

Now, an ordered pair (x, y) can be considered to be the "coordinates" of a cell in this array:

$$\text{cell } (x, y) \text{ is located in row } x, \text{ column } y.$$

Therefore, let each of the 16 ordered pairs (x, y) be converted into an ordered triple (x, y, z) by appending the number in cell (x, y); for example

$$(2,3) \to (2,3,2), \quad \text{and} \quad (4,2) \to (4,2,3).$$

Assembling the resulting triples in columns, then, gives the desired array of 16 prescriptions (A, B, C): (a so-called orthogonal array)

A	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
B	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
C	1	2	3	4	4	1	2	3	3	4	1	2	2	3	4	1.

It is easy to check directly that each ordered pair occurs exactly once in each pair of rows.

Therefore, if two numbers of the combination (a, b, c) belong to $(1,2,3,4)$, one of these 16 tries will open the safe. Otherwise, some two of (a, b, c) must come from $(5,6,7,8)$, and the 3×16 array corresponding to these numbers cannot fail to provide a successful trial. This second array is obtained from the array for $(1,2,3,4)$ simply by adding 4 to each entry. Thus the safe can certainly be opened in 32 or fewer tries.

In order to establish that 32 is the minimum, we need to show that, no matter what 31 attempts might be made, there is some combination (a, b, c) which has been missed completely, that is, none of whose pairs (a, b), (b, c), or (c, a) have been tested. To this end, let's keep track of our attempts in an 8×8 array R by entering the number c of the trial (a, b, c) in the cell with coordinates (a, b), that is, in row a and column b. Thus, if the cell in row 2 and column 5 holds the number 6, it means that one of our attempts was the combination $(2,5,6)$. We note that if we were wasteful enough to also try the combination $(2,5,3)$,

then the number 3 would also be entered in cell (2, 5) along with the 6. Suppose, then, that we have recorded 31 trials in the array R. Then one of four cases must occur.

(i) Suppose some row or column contains exactly two entries; for definiteness, suppose row 3 contains only 4 and 7, as shown. These entries could conceivably occur in the same cell but, in any case, there are at least 6 columns which do not have an entry in row 3. Now, each of these 6 columns could not have as many as 5 entries, for that would require $6 \cdot 5 = 30$ entries and there are only 29 other trials recorded in R. It follows, then, that one of these columns must hold not more than 4 entries; suppose column 7 has only the entries 8, 1, 3, 5, as shown. Altogether, then, the entries in row 3 and column 7 account for only 6 of the 8 numbers $(1, 2, 3, 4, 5, 6, 7, 8)$, namely the numbers $\{4, 7, 8, 1, 3, 5\}$. The number 6 (and 2) does not occur anywhere in row 3 or column 7. Therefore, if the correct combination happened to be $(3, 7, 6)$, our trials would have missed it completely:

— if we had tried any combination $(3, 7, c)$, then the number c would have been entered in cell $(3, 7)$; but cell $(3, 7)$ is empty;

— if any $(3, b, 6)$ had been tried, then 6 would have been entered somewhere in row 3 (in column b), which it wasn't;

— if any $(a, 7, 6)$ had been tested, then 6 would occur somewhere in column 7, which it doesn't.

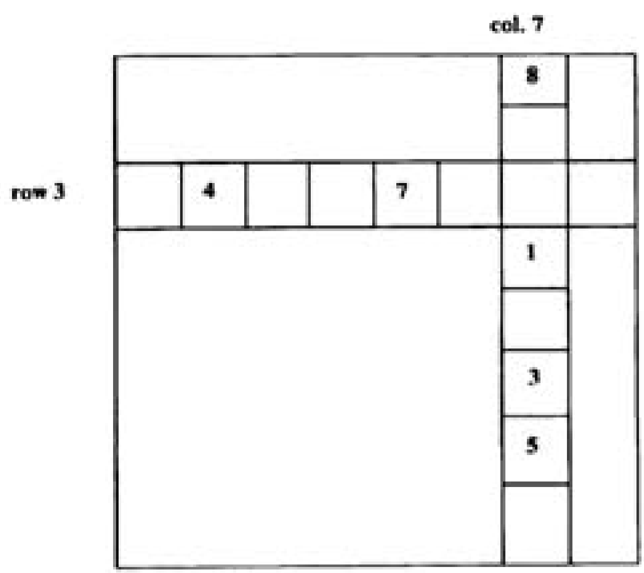


FIGURE 31

This argument goes through with at least equal force in the cases in which a row or column contains exactly one entry or no entry at all. Since 32 trials would be needed to supply the 8 rows of R with as many as 4 entries each, our 31 attempts must leave some row with no more than 3 entries. It remains, therefore, only to consider the case of a row which contains exactly 3 entries.

(ii) Suppose row 1 contains 3 entries in columns 1, 2, and 3, as shown. If any of these columns 1, 2, 3 were to contain exactly one or exactly two entries, our claim would be substantiated by one of the previous cases. Suppose, then, that each of these columns has at least 3 entries, for a total of at least 9 in these columns altogether. The other 5 columns, then, cannot each have as many as 5 entries, for there are only 22 or fewer trials recorded in them. Thus one of these other columns must have 4 or fewer entries and our conclusion follows as above.

If 2 or more of the 3 entries in row 1 occur in the same cell, our argument is reinforced:
three entries in the same cell implies not more than 28 in the other 7 columns; 3 entries in just 2 cells implies at least 6 in their 2 columns, leaving not more than 25 in the other 6 columns; in any case, one of the other columns must have not more than 4 entries, and this column and row 1 between them do not contain all 8 numbers.

Therefore there is always at least one combination that remains completely untested by any series of 31 attempts to open the safe, and it follows that the minimum is indeed 32.

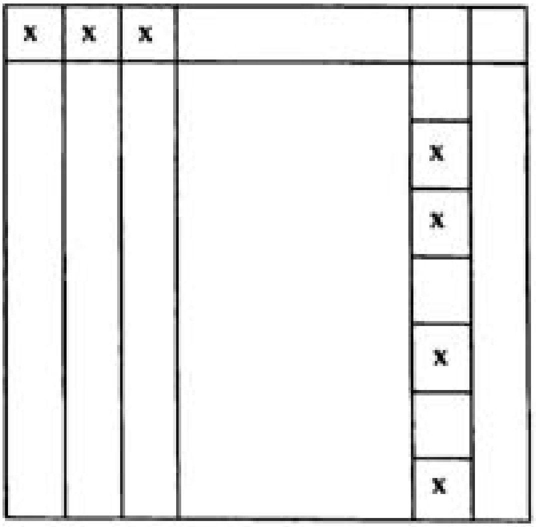


FIGURE 32

3. Proposer Unspecified

(*Crux Mathematicorum*, 1988, 260)

Let S be the set of all binary sequences of length 7:

$$S = \{0000000, 0000001, \ldots, 1111111\}.$$

Let the number of places in which two sequences differ be called the *distance* between them, and suppose that T is a subset of S in which the distance between each two members is at least 3.

Prove that T cannot have more than 16 members, and construct such a subset T which has 16 members.

Solution

Clearly there are 7 sequences which differ from a given sequence A in *exactly one* place; for example, the 7 sequences that are distance one from $A = 0000000$ are 1000000, 0100000, 0010000, 0001000, 0000100, 0000010, 0000001. Thus the sequences of S go together in subsets of 8 consisting of a sequence A and its 7 associates at distance 1; let the subset for A be denoted by $\bar{A}$.

Suppose T has n elements $\{A_1, A_2, \ldots, A_n\}$, and consider their associated subsets $\{\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n\}$. It is easy to see that no two of these subsets overlap, for if a sequence B were to belong to both $\overline{A}_i$ and $\overline{A}_j$, we would have the contradiction that the distance between A_i and A_j could not exceed 2: in this case,

> changing not more than one place in A_i gives B (B could be A_i itself) and changing not more than one place in B gives A_j.

Altogether, then, the n members of T are associated with $8n$ different sequences in S. Since S only has a total of $2^7 = 128$ sequences, then

$$8n \le 128, \quad \text{and} \quad n \le 16.$$

This is the easy part of the question. How to construct such a 16-member subset T is far from obvious; it is very easy to become hopelessly lost in one's initial attempts.

This problem, however, is encountered early in the subject of coding theory and can be solved very nicely by using the so-called Fano Plane (Figure 33). Let the numbers 1, 2, 3, 4, 5, 6, 7 represent the 7 places in a sequence of S. The lines in this figure (one of which is curved, namely 2 4 6) show us how to construct, from 0000000, a set of 7 sequences whose distances from 0000000 are all 3 and whose distances from each other are always 4. Each line contains three of the

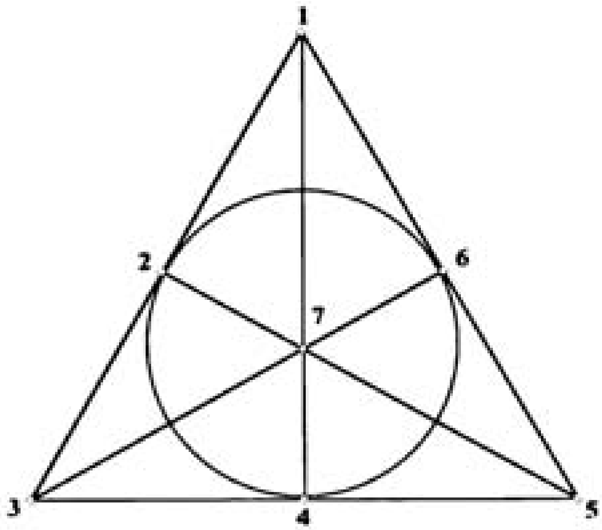


FIGURE 33

numbers 1, 2, 3, 4, 5, 6, 7, and all we need to do is to put 1's in these places and 0's everywhere else:

X

0	0	0	0	0	0	0
1	1	1	0	0	0	0
0	0	1	1	1	0	0
1	0	0	0	1	1	0
1	0	0	1	0	0	1
0	0	1	0	0	1	1
0	1	0	0	1	0	1
0	1	0	1	0	1	0.

This set X can thus contribute 8 sequences to our desired subset T. For the remaining 8 members of T, we can use the complements of these, that is, the sequences obtained by interchanging the 0's and 1's.

Y

1	1	1	1	1	1	1
0	0	0	1	1	1	1
1	1	0	0	0	1	1
0	1	1	1	0	0	1
0	1	1	0	1	1	0
1	1	0	1	1	0	0
1	0	1	1	0	1	0
1	0	1	0	1	0	1.

The distance between two members of Y is therefore automatically 3 or 4, as the distance between their antecedents is 3 or 4; the only question is whether some member of X is closer than 3 to some member of Y. There doesn't seem to be any way around a great many direct comparisons, and you may prefer simply to check each member of X with each member of Y and be done with it. However, such investigations can be presented nicely in the following form.

The complementary pair $P = 0000000$ and $P' = 1111111$ are best considered separately. Since each member of Y has at least 4 1's, the distance from P to a member of Y is at least 4; similarly P' is at least a distance 4 from any member of X. Also, the distance between any sequence and its complement is 7, and so it remains only to check the distance between a member A of $\{X - P\}$ and a sequence B' from $\{Y - P'\}$ which is not the complement of A.

A direct check reveals that the distances between A and any other member of its own section $\{X - P\}$ is always exactly 4; that is to say, two members of $\{X - P\}$ *agree in exactly 3 places* and differ in 4. Similarly for any two members of $\{Y - P'\}$. Now consider the complement A' of A. A' belongs to Y, and we have just noted that B' agrees with A' in exactly 3 places u, v, w. Since all the

$$\begin{array}{ll} X & Y \\ A : \cdots u'v'w' & A' : \cdots uvw \\ & B' : \cdots uvw \end{array}$$

places in A' need reversing to give A, then A must differ from B' in exactly these three places u, v, and w, and the solution is complete.

4. Proposed by the USA

(*Crux Mathematicorum*, 1988, 259)

A number of signal lights are equally spaced along a one-way railroad track, labelled in order $0, 1, 2, \ldots, n, n \geq 2$. As a safety rule, a train is not allowed to pass a signal if any other train is in motion on the length of track between it and the following signal. However, there is no limit to the number of trains that can be parked motionless at a signal, one behind the other, waiting their turn to cross the next section of track (we assume these are point-trains and have zero length).

A series of freight trains must be driven from signal 0 to signal n. Each train travels at a distinct but constant speed at all times when it is not blocked by the safety rule. Show that, regardless of the order in which the trains might be arranged, the same time will elapse between the first train's departure from signal 0 and the last train's arrival at signal n.

Solution

The progress of the trains can be charted nicely on a time-distance graph as illustrated (Figure 34) — the signals are marked at equal intervals up the y-axis and the time is tracked along the x-axis. Since the speed of a train is constant, its progress across a section between consecutive signals is given by a straight segment, and the steeper its slope the faster the train. Clearly the first train is never held up and goes straight across all the sections without delay; but the course of a later train might well be given by a zig-zag path in which the parts parallel to the x-axis represent time spent waiting for the next section of track to clear. The key to the problem, however, is the fact that the slowest train, wherever it might occur in the pack, never has to wait at any signal; actually this is an immediate corollary of the more general result:

> any train which is slower than all preceding trains is never kept waiting at any signal.

We proceed by induction.

We have observed that the first train is never held up and so the claim is valid for the rth train when $r = 1$. Suppose, for some $r \geq 1$, that the rth train is never delayed, implying a perfectly straight locus across our graph, and that the next train which is slower than the rth train is the sth one. Now, there may be a number of trains in between these two, as shown in the figures. Since the sth train is the next one that is slower than the rth train, all the intervening trains must be faster than the rth train. In this case, the graphs of trains $r, r+1, r+2, \ldots$ up to but not including the sth train, yield a configuration in which the sections between consecutive signals are all the same — it's just the section between the start and the first level repeated at every level. Thus, for example, in Figure 34, the segment GH along level 2 is equal to CE on level 1; since GH is also clearly equal to DF on level 1, it follows that

$$CD = CE + ED = GH + ED = DF + ED = EF.$$

In Figure 34, then, we observe straightforwardly that $AB = CD = EF$. (See the derivation just below Figure 34.)

Now, train s is slower that train r, and therefore its locus across a section of track will be given by a segment having a smaller slope than the slope AC = slope IF of train r. It follows, then, that the locus of s must cross level 1 at a point J that is later than the time F, indicating that train s is not delayed at level 1, and consequently not at any later level either.

Since Figure 34 is not quite general, a second typical situation is pictured in Figure 35, which leads to the same conclusion, as argued below the figure, where it is established that $AB = IJ < IN$.

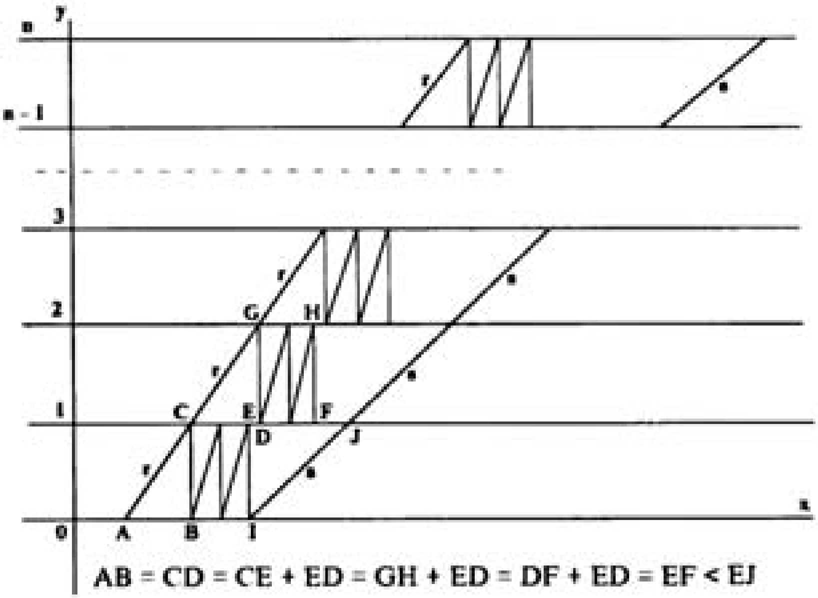


FIGURE 34

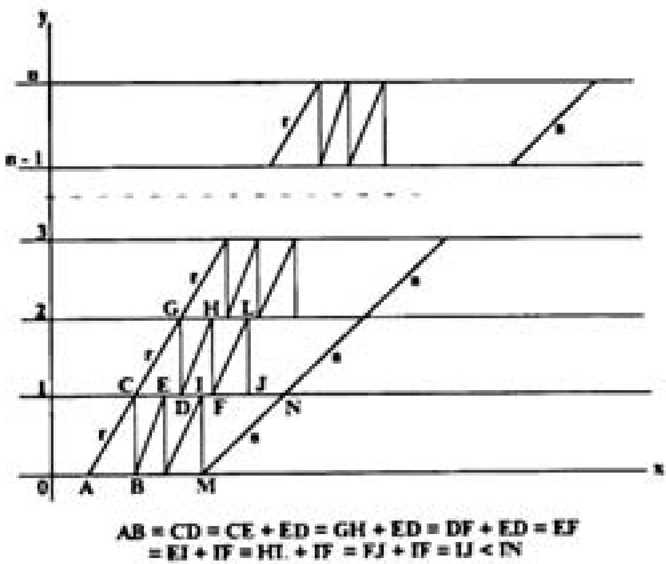


FIGURE 35

To finish the solution, let the k trains be arranged in any order and suppose that t_i is the time it takes train i to cross one section of track. It is clear from Figure 36 that the total time T taken for the entire enterprise can be traced along

the time-axis, and if t_s is the time for the slowest train, we have simply that

$$T = \sum_{i=1}^{k} t_i + (n-1)t_s,$$

which is clearly a constant.

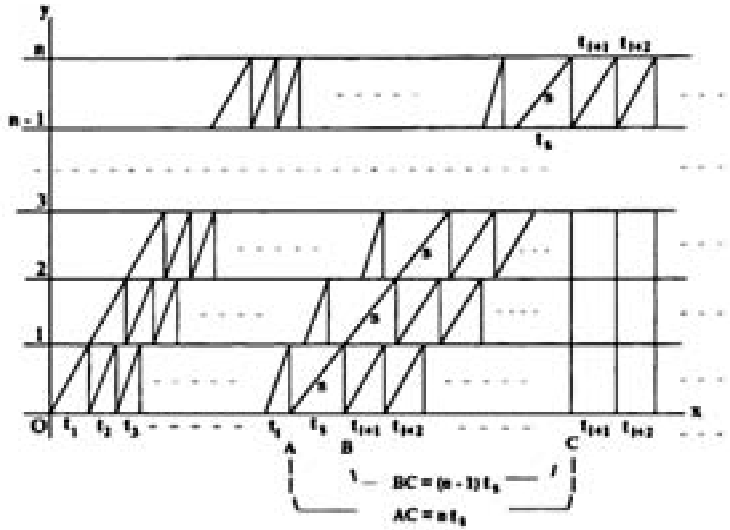


FIGURE 36

From the 1988 AIME Examination

(*Crux Mathematicorum*, 1988, 99)

Year after year I am amazed at the ingenuity the composers manage to pack into the American Invitational Mathematics Examination. The problems are of remarkable diversity and quality and bring much pleasure to the entire mathematical community. Since only three hours are allowed for 15 questions, the problems are not as difficult as olympiad problems. At their own level, however, they are often delightfully ingenious.

			*	
	74			
				186
		103		
0				

FIGURE 37

Problem 1

It is possible to place positive integers into the 21 vacant cells of the given 5×5 array shown in Figure 37 so that the numbers in each row and in each column form an arithmetic progression. Determine the number that must occupy the cell marked $*$.

4d			*	
3d	74			
2d	y			186
d	x	103		
0				

FIGURE 38(a)

52	82	112	142	
39	74			
26	66			186
13	58	103		
0				

FIGURE 38(b)

Solution

The first column, from bottom to top, must be $(0, d, 2d, 3d, 4d)$ (Figure 38(a)). In terms of d, the entries x and y in figure (a) can be determined as the arithmetic means between their neighbors as follows:

$$\text{(i)}\ x = \frac{d+103}{2},$$

and then

$$\text{(ii)}\ y = \frac{x+74}{2} = \frac{d+251}{4}.$$

Accordingly, the common difference of the progression in the third row is

$$D = y - 2d = \frac{251-7d}{4}.$$

Since the 186 at the end of this row is simply $2d + 4D$, we have

$$2d + (251 - 7d) = 186,$$
$$65 = 5d$$

and

$$d = 13.$$

This gives $x = 58$ and $y = 66$, from which the first two columns can be determined, thus setting the progression along the top row at $52, 82, \ldots$. Thus the desired entry is 142, as shown in Figure 38(b).

Problem 2

The faces of a convex polyhedron are 12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments which join a pair of vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or across a face?

Solution

As implied in the problem, the number of segments that pass through the interior of the polyhedron is simply

$D =$ the total number of segments which join a pair of vertices
$-$ (the number of edges) $-$ (the number of face diagonals).

Each of these quantities is easily calculated as follows.

(i) *The Edges.* Each square has 4 edges, each hexagon 6 and each octagon 8, for a total of $12 \cdot 4 + 6 \cdot 8 + 8 \cdot 6 = 144$ edges. However, since each edge lies in the two faces it borders, each edge is counted twice in this total, implying that the actual number of edges is only 72.

(ii) *The Face Diagonals.* Each square has 2 diagonals, each hexagon $\binom{6}{2} - 6 = 9$ (altogether there are $\binom{6}{2} = 15$ segments, 6 of which are edges), and each octagon has $\binom{8}{2} - 8 = 20$, for a total of

$$12 \cdot 2 + 8 \cdot 9 + 6 \cdot 20 = 216,$$

in which each diagonal is properly counted just once.

(iii) *The Vertices.* Finally, among the faces, the total number of vertices is

$$12 \cdot 4 + 8 \cdot 6 + 6 \cdot 8 = 144.$$

However, there are really only $\frac{144}{3} = 48$ vertices in the polyhedron because the vertices of the faces come together in 3's at the vertices of the polyhedron.

Therefore the desired number of interior diagonals is

$$\begin{aligned} D &= \binom{48}{2} - 72 - 216 \\ &= 24 \cdot 47 - 72 - 216 = 24(47 - 3 - 9) \\ &= 24 \cdot 35 = 840. \end{aligned}$$

Problem 3

Find a, if a and b are integers such that $x^2 - x - 1$ is a factor of

$$ax^{17} + bx^{16} + 1.$$

Solution

By the factor theorem, $x - c$ is a factor of an integral polynomial $f(x)$ if and only if $f(c) = 0$. If the roots of $x^2 - x - 1 = 0$ are α and β, then $x^2 - x - 1 = (x - \alpha)(x - \beta)$, and if $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$, then so is each of $x - \alpha$ and $x - \beta$, and we would have

$$a\alpha^{17} + b\alpha^{16} + 1 = 0$$

and

$$a\beta^{17} + b\beta^{16} + 1 = 0.$$

Solving for a, we proceed

$$a\alpha^{17}\beta^{16} + b\alpha^{16}\beta^{16} + \beta^{16} = 0,$$
$$a\alpha^{16}\beta^{17} + b\alpha^{16}\beta^{16} + \alpha^{16} = 0,$$

and subtraction gives

$$a\alpha^{16}\beta^{16}(\alpha - \beta) + \beta^{16} - \alpha^{16} = 0,$$

from which

$$a = \frac{\alpha^{16} - \beta^{16}}{\alpha^{16}\beta^{16}(\alpha - \beta)}.$$

Since α and β are the roots of $x^2 - x - 1 = 0$, their product $\alpha\beta = -1$, and so this reduces to

$$a = \frac{\alpha^{16} - \beta^{16}}{\alpha - \beta}.$$

The value of $\alpha - \beta$ is easily calculated to be $\sqrt{5}$, but the numerator $\alpha^{16} - \beta^{16}$ is a formidable expression in view of the irrational values of α and β:

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

It certainly pays, therefore, to know that α and β are the numbers involved in Binet's formula for the Fibonacci numbers:

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Thus the required number is none other than f_{16}, which is easily found to be 987 by working out the first 16 terms of the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, <u>987</u>, ...

An Unused Bulgarian Problem on the Medial Triangle and the Gergonne Triangle

The triangle determined by the midpoints A', B', C' of the sides of $\triangle ABC$ is called its *medial* triangle. Clearly each side of the medial triangle is parallel to the opposite side of $\triangle ABC$, making the triangles equiangular:

$$\measuredangle A = \measuredangle A', \quad \measuredangle B = \measuredangle B', \quad \measuredangle C = \measuredangle C'.$$

It is an important property that corresponding angle bisectors in a triangle and its medial triangle are always parallel; for example,

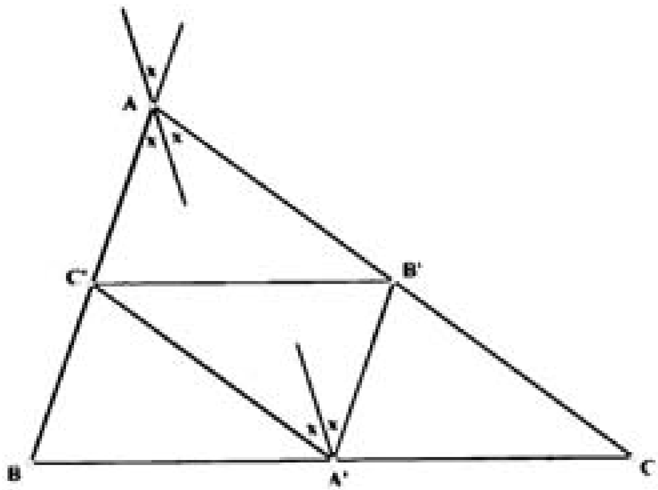


FIGURE 39

> the bisectors of the equal angles at A and A' are equally inclined to the common direction of AB and $A'B'$.

It might come as a mild surprise that the three lines from the vertices of a triangle ABC to the points of contact on the opposite sides that are determined by the incircle are always concurrent at a point G called its Gergonne point after the French geometer Joseph Gergonne (1771–1859) (Figure 40). Since the locations of these points of contact D, E, F are not known in readily usable terms, it might appear that this result will be difficult to prove. While it is readily proved by Ceva's theorem, it can also be established very neatly as follows.

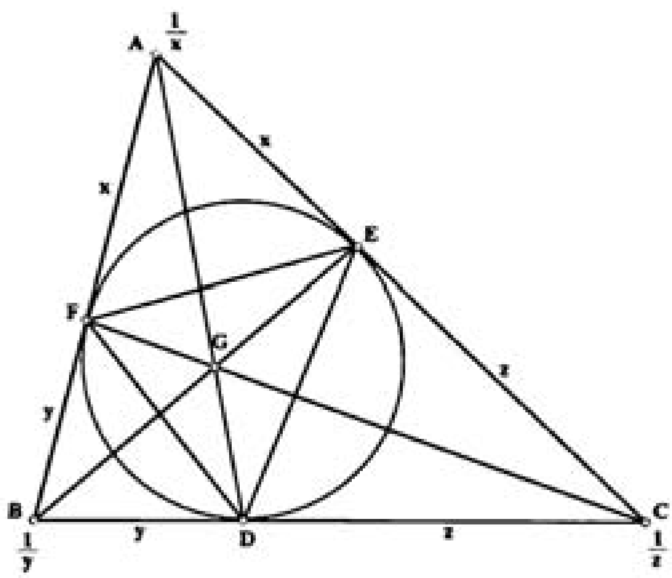


FIGURE 40

Let the equal tangents to the incircle from the vertices have lengths x, y, and z, and let masses of $1/x$, $1/y$, and $1/z$ be suspended, respectively, at A, B, and C, as shown. Then clearly F is the center of gravity of the masses at A and $B \left(\frac{1}{x} \cdot x = \frac{1}{y} \cdot y\right)$, implying that the system of masses is equivalent to a mass of $\left(\frac{1}{x} + \frac{1}{y}\right)$ at F and $1/z$ at C. The center of gravity of the whole system, then, must lie somewhere on FC. But the same is true of BE and AD, and the conclusion follows.

Although I cannot claim that it is common practice to do so in mathematical literature, let us call the triangle DEF determined by these points of contact with the incircle the *Gergonne* triangle of $\triangle ABC$. Accordingly, a beautiful problem which was submitted a few years ago by Bulgaria for an international olympiad, but not used, may be stated in the following simple terms.

> In any triangle, prove that lines from the vertices of the medial triangle, which are respectively perpendicular to the opposite sides of the Gergonne triangle, are always *concurrent*.

On the face of it, there doesn't seem to be any special connection between the medial triangle and the Gergonne triangle, and so we might expect the solution to hinge on some subtle, complicated, and possibly not very interesting relationship. The following transparent solution, due to J. T. Groenman, of Arnhem, The Netherlands, is therefore a most delightful surprise (reported in *Crux Mathematicorum*, 1987, 75).

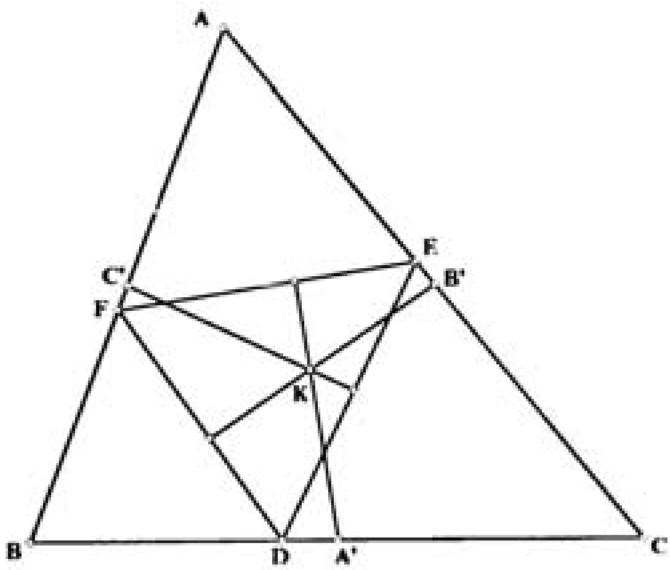


FIGURE 41

The equal tangents to the incircle AF and AE make $\triangle AFE$ isosceles (Figure 42), and therefore the bisector AL of its vertical angle is perpendicular to the base FE. That is to say, any perpendicular to the side FE of the Gergonne triangle lies in the same direction as AL.

But we saw above that the bisector of $\measuredangle A'$ in the medial triangle is parallel to the bisector of $\measuredangle A$, and therefore the bisector of $\measuredangle A'$ must in fact be the very perpendicular in question from A' to FE. Hence the three perpendiculars are simply the angle bisectors of the medial triangle, and therefore they meet at the incenter I' of the medial triangle.

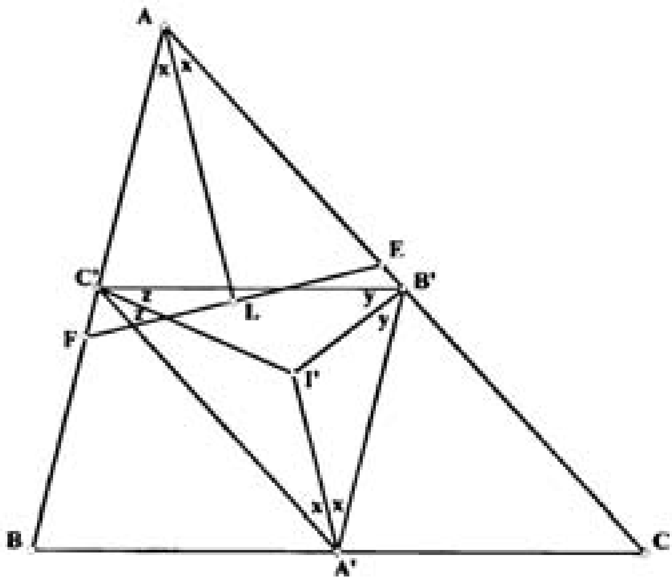


FIGURE 42

Two Solutions by John Morvay from the 1982 Leningrad High School Olympiad

(*Crux Mathematicorum*, 1988, 107)

John Morvay is from Dallas, Texas, and he has had many outstanding solutions published in *Crux Mathematicorum*.

Problem 1

The first four terms of an infinite sequence S of decimal digits are $1, 9, 8, 2$, and succeeding terms are given by the final digit in the sum of the four immediately preceding terms. Thus S begins

$$1, 9, 8, 2, 0, 9, 9, 0, 8, 6, 3, 7, 4, \ldots.$$

Do the digits $3, 0, 4, 4$, ever come up consecutively in S?

Solution

The thing to notice is that this sequence is completely determined by *any* four consecutive terms. For example, not only do the four terms $8, 2, 0, 9$ in positions 3 to 6 lead forward to all later terms, but they also generate all preceding terms, too. Clearly

$$y, x, 8, 2, 0, 9$$

implies that the last digit of $x + 8 + 2 + 0$ is 9, forcing x to be 9, after which the last digit of $y + 9 + 8 + 2$ must be 0, making $y = 1$. In fact, this sequence S is merely one branch of a sequence T that extends infinitely far in both directions;

and four consecutive terms taken anywhere along T determine the whole sequence.

Extending backwards from S a little way, we find

$$\ldots, 3, 0, 4, 4, \overleftarrow{1, 9}, 8, 2, \ldots,$$

revealing that the object of our investigations, namely the four consecutive terms 3, 0, 4, 4, certainly occur in the full sequence T. Consequently, if T is periodic throughout its entire length, then 3, 0, 4, 4, would occur infinitely often in each of its branches.

Accordingly, let T be partitioned into abutting blocks of four consecutive digits on each side of the given block $\underline{1, 9, 8, 2}$.

$$\overleftarrow{\ \ T\ \ }\!\!\rightarrow$$

$$\ldots \underline{3, 0, 4, 4} \quad \underline{1, 9, 8, 2} \quad \underline{0, 9, 9, 0} \quad \underline{8, 6, 3, 7} \ldots$$

Since there are only 10^4 different blocks possible in the decimal system, some block $\underline{a, b, c, d}$ must arise in T for a second time, say as the mth and nth blocks counting from the given block $\underline{1, 9, 8, 2}$, m and/or n possibly negative.

$$\ldots \underline{3, 0, 4, 4}, \quad \underline{1, 9, 8, 2}, \ldots \underset{(m)}{\underline{a, b, c, d}} \ldots \underset{(n)}{\underline{a, b, c, d}} \ldots$$

Since the entire sequence T can be generated from any block, the blocks which immediately precede the repeated occurrences of $\underline{a}, \underline{b}, \underline{c}, \underline{d}$, namely the blocks in positions $m - 1$ and $n - 1$, must also be the same as each other. In turn, these force the block in position $m - 2$ to be a copy of the one in position $n - 2$, and so on backward, and similarly forward, throughout the entire length of T. Thus T *is* periodic alright, and we conclude that S does indeed contain 3, 0, 4, 4, in fact infinitely often.

Problem 2

Each cell in a 5×41 rectangular grid is colored either red or blue. Prove that some 3 rows and some 3 columns must intersect in 9 cells of the same color.

Solution

Since only 2 colors are used, the pigeonhole principle implies that, in a column of 5 cells, one of the colors must occur in at least 3 of the cells. Let this "majority" color be noted for each of the 41 columns.

Again, since there are only 2 colors, among these 41 majority colors, one of the colors must be attached to at least 21 of the columns. Let us disregard all

the other columns except for any 21 of them which have the same majority color, say red.

In each of these 21 columns, note any 3 of its rows which contain red cells (there might be more than 3 to choose from). Disregarding all other cells in these columns, we are reduced to 21 columns, each containing 3 red cells in some subset of 3 of its 5 rows.

Now, there are only $\binom{5}{3} = 10$ possible subsets of 3 rows that can be selected from a column of 5 rows ($\{(1,2,3),(1,2,4),\ldots,(3,4,5)\}$). Thus, in a collection of 21 columns, at least one of the 10 possible subsets (a,b,c) must occur in 3 or more of the columns $(c_1, c_2, c_3, \ldots)$. (This follows from the more general form of the pigeonhole principle: if $kn + 1$ objects are distributed into n boxes, some box must contain at least $k + 1$ of the objects; otherwise the total count could not amount to more than kn, a contradiction.)

$$\begin{array}{cccc} & c_1 & c_2 & c_3 \\ a & \bullet & \bullet & \bullet \\ \hdashline & & & \\ \hdashline b & \bullet & \bullet & \bullet \\ c & \bullet & \bullet & \bullet \end{array}$$

Thus the 9 cells of intersection of these rows and columns are all red.

Exercise

To what dimensions would the array have to be extended in order to guarantee a monochromatic subgrid of size 4×4 if 3 colors are used?

(Incidentally, and perhaps surprisingly, the larger dimension is the year my father was born.)

Solution

To insure a monochromatic grid of size $a \times a$ when c colors are permitted, the array needs to be enlarged to have dimensions

$$c(a-1) + 1 \text{ by } c\left[(a-1)\binom{c(a-1)+1}{a}\right] + 1.$$

For a 4×4 grid with 3 colors, then, a 10×1891 array is required; for a 5×5 grid with 4 colors, we need a 17×99009 array.

Two Solutions by Ed Doolittle

(*Crux Mathematicorum*, 1988, 70, 71)

Now let's turn to two problems that were left over from the 1985 International Olympiad. These beautiful solutions are due to Ed Doolittle of the University of Toronto. (The first problem was also solved by that ingenious problem-solver Daniel Ropp of Washington State University, St. Louis, Missouri.)

1. Proposed by Rumania

Since $\sqrt{2}$ is an irrational number, the value of $n\sqrt{2}$ is never a whole number for any positive integer n. In this problem, we are concerned with the sequence of integers S that is obtained from the values of $n\sqrt{2}$ by dropping their fractional parts, that is, the sequence of integer parts $[n\sqrt{2}], n = 1, 2, 3, \ldots$.

The first three terms of S are

$$[\sqrt{2}] = 1, \quad [2\sqrt{2}] = [2.8\ldots] = 2, \quad [3\sqrt{2}] = [4.24\ldots] = 4,$$

each of which, we might note, is a power of 2. Again, the sixth term is $[6\sqrt{2}] = [8.48\ldots] = 8$, and the twelfth is $[12\sqrt{2}] = [16.9\ldots] = 16$, giving two more powers of 2. Sometimes a power of 2 is skipped over, as in the case of 64:

$$[45\sqrt{2}] = [63.6\ldots] = 63, \quad \text{and} \quad [46\sqrt{2}] = [65.05\ldots] = 65.$$

Prove, however, that S contains an infinite number of powers of 2.

Solution

Doolittle's solution is nothing short of brilliant! He begins by noting that, when expressed as a binary decimal, the value of $\sqrt{2}$ contains an infinite number of 1's:

$$\sqrt{2} = 1.01101\ldots;$$

if this were not the case, then, beyond some point, all the digits would be the only other possibility, namely 0, resulting in a terminating decimal and implying that $\sqrt{2}$ is rational. Now, in the binary scale, multiplication by the base 2 moves the decimal point one place to the right:

$$2\sqrt{2} = 10.1101\ldots, \quad 2^2\sqrt{2} = 101.101\ldots,$$

and so forth. Since $\sqrt{2}$ contains an infinite number of 1's, there is an infinity of powers 2^n which carry the decimal point of $2^n\sqrt{2}$ just to the left of a 1, as in $2^2\sqrt{2} = 101.101\ldots$, in which case the *fractional part* of $2^n\sqrt{2}$ is an expression like $.101\ldots$, beginning with a 1 right after the decimal point and, because there are always other 1's later in the expression, has a value that exceeds

$$.10000000\ldots = .1\dot{0} = \frac{1}{2}.$$

Denoting the fractional part of x by $\{x\}$, we have established, then, that there is an infinite set of values of n, $A = (1, 2, \ldots)$, which satisfy

$$\{2^n\sqrt{2}\} > \frac{1}{2}.$$

Now, it's not really $\frac{1}{2}$ that we are interested in—it's $1 - \frac{1}{\sqrt{2}}$. Since this is less than $\frac{1}{2}$, every value of n in the infinite set A also satisfies

$$\{2^n\sqrt{2}\} > 1 - \frac{1}{\sqrt{2}}.$$

In each case, then, we have

$$\frac{1}{\sqrt{2}} > 1 - \{2^n\sqrt{2}\},$$

and, reversing the sides,

$$0 < (1 - \{2^n\sqrt{2}\})\sqrt{2} < 1.$$

Clearly, adding a quantity which is between 0 and 1 to an *integer m* does not carry you to the next integer and so does not alter the integer part of the number:

for $0 < x < 1$, $[m + x]$ is still the integer m.

Therefore, adding $(1 - \{2^n\sqrt{2}\})\sqrt{2}$ to 2^{n+1} does not alter the integer part and we have

$$\left[2^{n+1} + (1 - \{2^n\sqrt{2}\})\sqrt{2}\right] = 2^{n+1}.$$

Since 2^{n+1} factors into $\sqrt{2}(2^n\sqrt{2})$, we can take out a common factor of $\sqrt{2}$ to give

$$\left[(2^n\sqrt{2} + 1 - \{2^n\sqrt{2}\})\sqrt{2}\right] = 2^{n+1},$$

and, slightly rearranged,

$$\left[(2^n\sqrt{2} - \{2^n\sqrt{2}\} + 1)\sqrt{2}\right] = 2^{n+1}.$$

Now, removing the fractional part from a number leaves just the integer part, and so

$$2^n\sqrt{2} - \{2^n\sqrt{2}\} = [2^n\sqrt{2}].$$

Hence we have

$$\left[([2^n\sqrt{2}] + 1)\sqrt{2}\right] = 2^{n+1}.$$

Finally, adding 1 to an integer part $[x]$ simply gives the integer part of the number $x + 1$:

$$[x] + 1 = [x + 1].$$

Thus

$$\left[[2^n\sqrt{2} + 1]\sqrt{2}\right] = 2^{n+1}.$$

That is to say, for the integer $k = [2^n\sqrt{2} + 1]$, we have

$$[k\sqrt{2}] = 2^{n+1}, \text{ a power of 2.}$$

Since different values of n clearly yield different values of k, the infinite set A generates the desired infinity of powers of 2 in the given sequence S.

2. A Geometric Construction

A Problem from Spain

As usual, let the circumcenter and orthocenter of a triangle be called O and H, respectively. Our problem is to construct a triangle ABC from the following

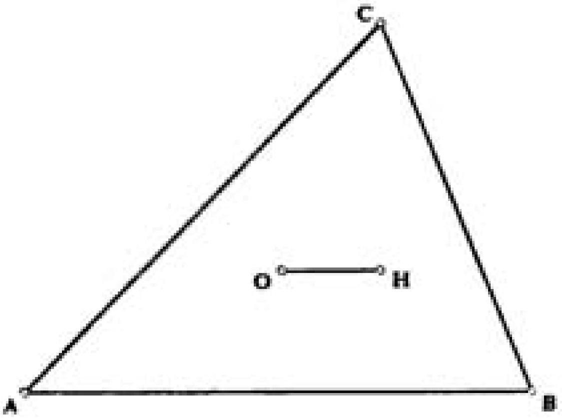


FIGURE 43

scanty information about AB and OH: all you are given is their lengths and the fact that they are parallel. (Of course, their relative *positions* are not given.)

The Circle of Apollonius

Doolittle's wonderful solution starts with any segment of the proper length for AB and then undertakes to locate the position of O relative to it, after which the construction is easily completed. An obvious locus through O is the perpendicular bisector of AB. The problem lies in finding a second constructible locus through O.

It turns out that it is not difficult to construct a point M with the property that O is twice as far from A as it is from M. Now, the great ancient Greek geometer Apollonius of Perga (262–195 B.C.) discovered that the locus of a

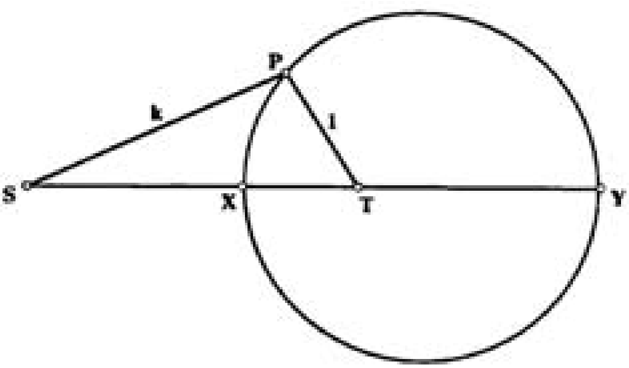


FIGURE 44

point P which moves so that it remains k times as far from a fixed point S as it is from a fixed point T is the circle with diameter XY, where X and Y are simply the points which divide ST internally and externally in the ratio $k:1$. Since straightedge and compasses suffice to divide a segment in a given ratio, a circle of Apollonius can readily be constructed for our points A and M and the ratio $2:1$ to provide the second locus through O. Let us digress briefly, then, to establish this useful theorem of Apollonius.

We need to show that any point P which is known to be k times as far from S as it is from T must lie on this circle, and conversely that every point on the circle is actually k times as far from S as it is from T. Let's take these parts in order.

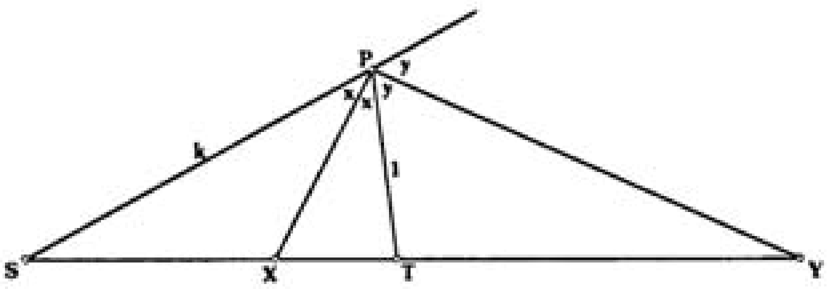


FIGURE 45

(i) $PS = k \cdot PT$

The proof in this section is based on the theorem that the internal and external bisectors at a vertex of a triangle strike the opposite side in the points which divide it internally and externally in the ratio of the other two sides of the triangle. On the strength of this, $PS = k \cdot PT$ immediately gives that

$$\frac{SX}{XT} = \frac{k}{1} \quad \text{and} \quad \frac{SY}{TY} = \frac{k}{1}.$$

In whatever way P might vary subject to $PS = k \cdot PT$, the bisector of $\measuredangle SPT$ always divides ST in the ratio $k:1$ and therefore always goes through the same point X. Similarly the external bisector always goes through Y, and it is clear that $\measuredangle XPY$ is always one-half a straight angle, i.e., a right angle, implying that P always lies on the circle on diameter XY.

(ii) *P is point of the circle on diameter XY*

Establishing the converse is a more interesting challenge. This time we know from their definitions that X and Y divide ST internally and externally in

the ratio of $k:1$, and we wish to show that $PS = k \cdot PT$. At least the right angle at P implies that the angles θ and ϕ at X and Y in $\triangle XPY$ add up to a right angle. If these angles are reproduced as corresponding angles by drawing TU and TV parallel to PX and PY, respectively, then $\measuredangle VTU$ is also a right angle.

These parallels also transmit the ratio $k:1$ to parts of the line SU as follows:

$$\frac{SX}{XT} = \frac{SP}{PU} = \frac{k}{1} \quad \text{and} \quad \frac{SY}{TY} = \frac{SP}{VP} = \frac{k}{1}.$$

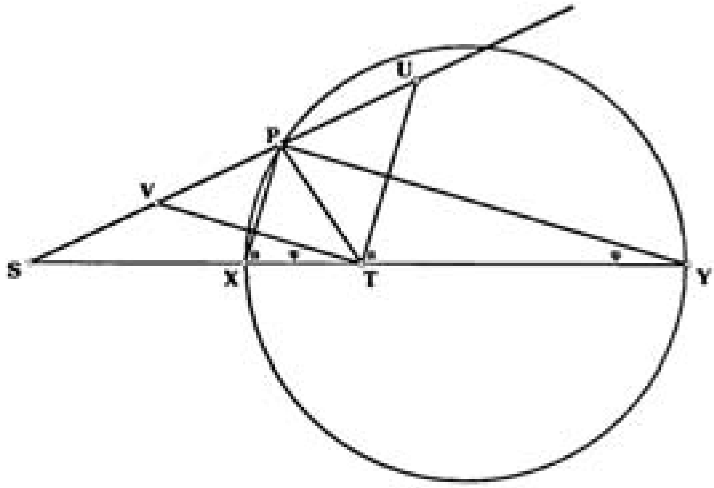


FIGURE 46

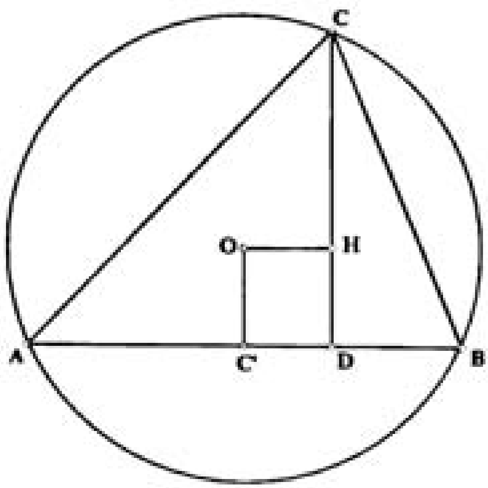


FIGURE 47

Consequently $\frac{SP}{PU} = \frac{SP}{VP}$, giving $PU = VP$, making P the midpoint of the hypotenuse of right-triangle VTU. Being the circumcenter of this triangle, P is the same distance from all three vertices, and we conclude that $PT = PU$. Finally, then,

$$\text{the known} \quad \frac{SP}{PU} = \frac{k}{1} \quad \text{gives the required} \quad \frac{SP}{PT} = \frac{k}{1}.$$

Now let's get on with the construction of triangle ABC.

Solution

Since the circumcenter O figures prominently in the given information, it certainly makes sense to consider the circumcircle. The altitude CH meets AB at right angles as does OC', the perpendicular bisector of AB (Figure 47). Therefore HD and OC' are parallel. Since OH and AB are also parallel, the quadrilateral $OC'DH$ is a rectangle, and therefore the length of $C'D$ is simply the given length OH. Thus, on any segment of the prescribed length AB, we can find the midpoint C' and lay off the given length OH from C' to get the point D. Thus, one constructible locus that goes through the third vertex C is the perpendicular to AB at this point D. As we shall see, as a second locus through C, fixing its position on this perpendicular, we will use the circumcircle itself. The problem reduces, then, to locating the center O. Of course, this requires two loci that pass through O. As already noted, these are the perpendicular bisector of AB and a certain circle of Apollonius. Finally, then, we come to the matter of locating this mysterious point M that we met in the preliminary discussion.

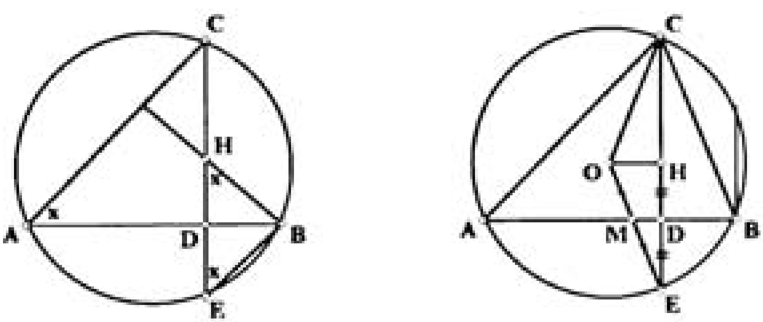


FIGURE 48

We need to use the engaging property that, if the altitude CD is extended to meet the circumcircle at E, then the foot D is the midpoint of HE, i.e., $HD = DE$ (Figure 48). This is established neatly by showing that $\triangle BHE$ is isosceles

because the base angles at H and E are equal (then the altitude BD bisects the base). To see this, we need only note that each of these base angles is equal to the angle A in the triangle:

> clearly the angles at A and E stand on the chord BC of the circle; and noting that the altitude BH is perpendicular to AC, the arms of the base angle at H are respectively perpendicular to the arms of the angle A (a quarter-turn about H would make the arms of the angle at H respectively parallel to the sides AC and AB).

Thus D is the midpoint of HE, and since DA is parallel to OH, in $\triangle OHE$ the side OE is crossed by AD at its own midpoint M (a line from the midpoint of one side of a triangle, parallel to a second side, bisects the third side). Therefore the distance from O to M is one-half the radius OE, while OA is a full radius of the circle. Hence O must lie on the circle of Apollonius with fixed points A and M and ratio $2:1$.

We mustn't overlook the fact that we haven't yet located the point M. This is easily remedied, however, by observing that DM, joining the midpoints of sides HE and OE in $\triangle OHE$, is one-half as long as the parallel side OH. Thus M can be constructed by laying off one-half the given length OH along DA from D, and the construction of $\triangle ABC$ is completed as planned above.

From the 1987 Spanish Olympiad

(*Crux Mathematicorum*, 1988, 132. An alternative solution is given in 1990, 72)

For each positive integer n, prove that the equation

$$P_n(x) = x^{n+2} - 2x + 1 = 0$$

has exactly one root c_n between 0 and 1, and determine $\lim_{n\to\infty} c_n$.

Solution

Since the sum of the coefficients is $1 - 2 + 1 = 0$, $x = 1$ is always one of the $n + 2$ roots and $x - 1$ is a factor of $x^{n+2} - 2x + 1$. The proposed root between 0 and 1, then, must come from the other factor of $x^{n+2} - 2x + 1$, suggesting that it might be worthwhile trying to find this other factor. However one might go about doing this (ordinary long division is one approach), it is easy to verify that the result is

$$x^{n+2} - 2x + 1 = (x - 1)(x^{n+1} + x^n + \cdots + x - 1)$$

(note the -1 at the end). The other $n + 1$ roots of our equation, then, are the roots of

$$Q_n(x) = x^{n+1} + x^n + \cdots + x - 1 = 0.$$

Now, $Q_n(0) = -1$, and, since $n + 1 \geq 2$, the value of $Q_n(1) \geq 1 + 1 - 1 > 0$. Because polynomials are continuous, then $Q_n(x) = 0$ for some x between 0 and 1 by the intermediate value theorem. Thus $P_n(x) = 0$ does have a root c_n between 0 and 1.

It is easy to see that the roots of $Q_n(x) = 0$ which lie between 0 and 1 are confined to this single value c_n. Each root x satisfies

$$x^{n+1} + x^n + \cdots + x = 1.$$

A second root y between 0 and 1 would also yield

$$y^{n+1} + y^n + \cdots + y = 1.$$

Now, if $y < x$, then $y^k < x^k$ throughout, and

$$y^{n+1} + y^n + \cdots + y < x^{n+1} + x^n \cdots + x = 1,$$

a contradiction. Similarly for $y > x$. We still need to show, however, that $y = x$ is also untenable, that is, that $x = c_n$ is not a multiple root.

Now, a multiple root is also a root of the equation obtained by differentiating $P_n(x) = 0$ (this is equivalent to the claim that the graph of $y = P_n(x)$ is tangent to the x-axis). If c_n were to be a multiple root, we would have both

$$c_n^{n+2} - 2c_n + 1 = 0,$$

and

$$(n+2)\,c_n^{n+1} - 2 = 0.$$

From the second equation we get

$$2 = (n+2)\,c_n^{n+1},$$

and substituting this in the first equation, we obtain

$$c_n^{n+2} - (n+2)\,c_n^{n+2} + 1 = 0,$$

giving

$$1 = (n+1)\,c_n^{n+2},$$

and

$$c_n^{n+2} = \frac{1}{n+1}.$$

Thus the first equation is just

$$\frac{1}{n+1} - 2c_n + 1 = 0,$$

implying

$$c_n = \frac{1}{2}\left(\frac{1}{n+1} + 1\right) = \frac{n+2}{2(n+1)},$$

and we also get

$$c_n^{n+2} = \left(\frac{n+2}{2(n+1)}\right)^{n+2}.$$

Equating these two values of c_n^{n+2}, we have

$$\left(\frac{n+2}{2(n+1)}\right)^{n+2} = \frac{1}{n+1},$$

$$(n+2)\left(\frac{n+2}{n+1}\right)^{n+1} = 2^{n+2},$$

and

$$\left(1+\frac{1}{n+1}\right)^{n+1} = \frac{2^{n+2}}{n+2}.$$

Now, it is well known that the left side is less than $e = 2.71\ldots$, the base of the natural logarithms, and therefore

$$\frac{2^{n+2}}{n+2} < 3,$$

implying $2^{n+2} < 3n+6$, which certainly fails for $n = 2, 3, \ldots$. Thus for $n \geq 2, x = c_n$ is not a multiple root of $P_n(x) = 0$. Solving $P_1(x) = 0$ separately, we quickly obtain three distinct roots and conclude that c_n is not a multiple root for all n:

$$P_1(x) = x^3 - 2x + 1 = (x-1)(x^2 + x - 1) = 0,$$

implying $x = 1$, and $\dfrac{-1 \pm \sqrt{5}}{2}$. Thus c_n is the one and only root of $P_n(x) = 0$ between 0 and 1 for all n.

Now, from

$$c_n^{n+2} - 2c_n + 1 = 0,$$

we get

$$c_n = \frac{1}{2} + \frac{1}{2}c_n^{n+2},$$

and

$$c_n > \frac{1}{2} \quad \text{for all } n.$$

It is tempting to finish things off with the argument that

$$c_n < 1 \quad \text{and} \quad c_n^{n+2} - 2c_n + 1 = 0$$

imply, as $n \to \infty$,

$$c_n^{n+2} \to 0 \quad \text{giving} \quad \lim c_n = \frac{1}{2}.$$

The trouble with this is that c_n *varies* with n and, for all we know, could conceivably be given by some function like

$$c_n = 1 - \frac{1}{n},$$

in which case c_n would approach 1. Clearly, then, this needs to be looked into more closely.

With an argument that was used earlier, it is not difficult to show that c_n decreases as n increases. Since c_n is a root of $Q_n(x) = 0$, we have

$$c_n^{n+1} + c_n^n + \cdots + c_n = 1,$$

and similarly that

$$c_{n+1}^{n+2} + c_{n+1}^{n+1} + \cdots + c_{n+1} = 1.$$

Now, if c_{n+1} were greater than or equal to c_n, then $c_{n+1}^k \geq c_n^k$ throughout, implying

$$\begin{aligned} c_{n+1}^{n+2} + (c_{n+1}^{n+1} + \cdots + c_{n+1}) &\geq c_{n+1}^{n+2} + (c_n^{n+1} + \cdots + c_n) \\ &= c_{n+1}^{n+2} + 1 > 1, \end{aligned}$$

a contradiction. It follows therefore that

$$c_{n+1} < c_n,$$

and that

$$c_n < c_1 \quad \text{for all } n > 1.$$

For $n > 1$, then, we have

$$\frac{1}{2} < c_n = \frac{1}{2} + \frac{1}{2} c_n^{n+2} < \frac{1}{2} + \frac{1}{2} c_1^{n+2},$$

where c_1 is a constant < 1, and we properly conclude that

$$\lim_{n\to\infty} c_n = \frac{1}{2}.$$

A Problem from Johann Walter

We are indebted to Johann Walter (Institut fur Mathematik, Aachen, Germany) for the following delightful problem.

> Prove that there are no *odd* positive integers x, y, and z which satisfy the Pythagorean relation
>
> $$(x+y)^2 + (x+z)^2 = (y+z)^2.$$

Dr. Walter supplied the following four solutions, which range from a simple direct approach to a somewhat sophisticated attack. In each case the conclusion is obtained by contradiction.

The Simplest Solution

In the event that some odd positive integers

$$x = 2a+1, \quad y = 2b+1, \quad z = 2c+1$$

were to satisfy the given Pythagorean relation, we would have

$$(2a+2b+2)^2 + (2a+2c+2)^2 = (2b+2c+2)^2$$

$$(a+b+1)^2 + (a+c+1)^2 = (b+c+1)^2$$

$$(a^2+b^2+1+2ab+2a+2b) + (a^2+c^2+1+2ac+2a+2c)$$
$$= b^2+c^2+1+2bc+2b+2c$$

$$2a^2+2ab+2ac+4a+1 = 2bc,$$

giving the contradiction of an odd number equal to an even number.

The Most Elegant Solution

(due to D. Kniepert, Aachen) If odd positive integers x, y, z satisfy

$$(x+y)^2 + (x+z)^2 = (y+z)^2,$$

then

$$(x^2 + 2xy + y^2) + (x^2 + 2xz + z^2) = y^2 + 2yz + z^2$$
$$x^2 + xy + xz = yz.$$

Adding yz to each side gives

$$(x+y)(x+y) = 2yz.$$

But this is impossible because each of $x + y$ and $x + z$ is even, making the left side divisible by 4, while the right side is divisible only by 2, since y and z are odd.

A Variation of Kniepart's Solution

(due to J. Walter) Continuing solution (ii) from the equation

$$x^2 + xy + xz = yz,$$

transposing xz and factoring gives

$$x(x+y) = z(y-x)$$

and

$$x \cdot \frac{x+y}{2} = z \cdot \frac{y-x}{2}.$$

Since $x + y$ and $y - x$ are both even, $\frac{x+y}{2}$ and $\frac{y-x}{2}$ are integers and this appears to be perfectly in order. However, this really equates an odd number and an even number, for $\frac{x+y}{2}$ and $\frac{y-x}{2}$ are of opposite party in view of the fact that their difference is the *odd* number x.

A More Sophisticated Approach

The basic theorem concerning Pythagorean triples states that (p, q, r) is a Pythagorean triple if and only if there exist positive integers k, m, and n, where m and n are relatively prime, one of m and n is even and the other odd, such that

$$p = 2kmn, \quad q = k(m^2 - n^2), \quad \text{and} \quad r = k(m^2 + n^2).$$

For $(p, q, r) = (x + y, x + z, y + z)$, we would have

$$x + y = 2kmn, \quad x + z = k(m^2 - n^2), \quad \text{and} \quad y + z = k(m^2 + n^2).$$

Now, if x, y, and z are all odd, then $x + z$ would be even, making

$$k(m^2 - n^2) \text{ an even integer.}$$

However, m and n are of opposite parity, and so $m^2 - n^2$ is odd. Thus it must be that k is even.

In this case,

$$\frac{1}{2}\left[(x + y) + (x + z) - (y + z)\right] = \frac{1}{2}\left[2kmn + k(m^2 - n^2) - k(m^2 + n^2)\right]$$
$$x = k(mn - n^2),$$

again equating an odd and an even integer.

Dr. Walter notes that the motivation for considering the expression on the left of this equation is the fact that

$$\frac{1}{2}\,[\text{the sum of the legs} - \text{the hypontenuse}]$$

is the formula for the radius of the inscribed circle in a Pythagorean triangle (i.e., a right-triangle having integral sides), and that such an inradius is known to be an integer.

This problem originates in J. Walter's paper "Über eine rationale Parameterdarstellung der Pythageräischen Zahlentripel" in *Der mathematische und Naturwissenschaftliche Unterricht*, vol. 44, pp. 451–456; for an English translation by the author, entitled "On a rational parametrization of the Pythagorean triples," contact the author at the Institut fur Mathematik, Templergraben 55, D-5100, Aachen, Germany.

From the 1987 Balkan Olympiad

(*Crux Mathematicorum*, 1988, 290)

Proposed by Bulgaria

Two circles K_1 and K_2, with centers O_1, O_2 and radii 1 and $\sqrt{2}$, respectively, intersect at A and B. AC is the chord of K_2 which is bisected by K_1. Determine the length of AC, given that O_1 and O_2 are 2 units apart.

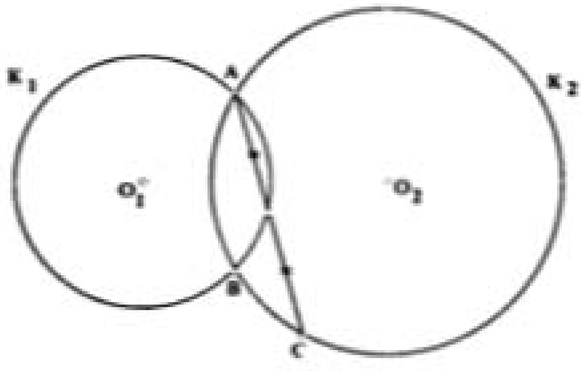


FIGURE 49

Solution

When two circles intersect, it is generally a good idea to give some thought to the diameters from one of the points of intersection, for the segment joining the

opposite ends of these diameters always goes through the other point of intersection; thus, in Figure 50, DE goes through B (the diameters subtend right angles at B). Also, the segment O_1O_2 between the centers joins the midpoints of the sides AD and AE in triangle ADE, and is therefore half as long as DE. Since $O_1O_2 = 2$, then $DE = 4$, and we know all three sides of triangle ADE:

$$AD = 2, \quad AE = 2\sqrt{2}, \quad DE = 4.$$

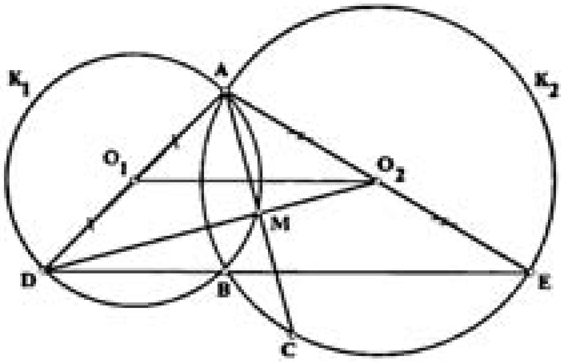


FIGURE 50

If DO_2 crosses K_1 at M, then angle AMD is a right angle, making the segment O_2M from the center of K_2 perpendicular to the chord AMC. Thus M is the midpoint of the chord and AC is in fact the chord whose length is required. Since $AC = 2AM$, let us determine the length of AM.

Since AM is an altitude of $\triangle ADO_2$, we have

$$\frac{1}{2} DO_2 \cdot AM = \text{the area of } \triangle ADO_2,$$

giving

$$AM = \frac{2\triangle ADO_2}{DO_2}.$$

Now, DO_2 is a median of $\triangle ADE$, and hence it bisects the area of $\triangle ADE$, and therefore

$$AM = \frac{\triangle ADE}{DO_2}.$$

The problem is essentially done now, for both the area and the length of a median are easily calculated when the three sides of a triangle are known.

(i) The semiperimeter of $\triangle ADE$ is $s = \frac{1}{2}(2 + 2\sqrt{2} + 4) = 3 + \sqrt{2}$, and hence, by Heron's formula,

$$\triangle ADE = \sqrt{(3+\sqrt{2})(1+\sqrt{2})(3-\sqrt{2})(-1+\sqrt{2})} = \sqrt{7}.$$

(ii) Applying the law of cosines to $\triangle ADE$, we get

$$\cos A = \frac{AD^2 + AE^2 - DE^2}{2AD \cdot AE}$$
$$= \frac{4+8-16}{8\sqrt{2}} = -\frac{1}{2\sqrt{2}}.$$

Similarly, applying the law of cosines to $\triangle ADO_2$, we obtain

$$DO_2^2 = AD^2 + AO_2^2 - 2AD \cdot AO_2 \cos A$$
$$= 4 + 2 + 4\sqrt{2} \cdot \frac{1}{2\sqrt{2}} = 8,$$

and

$$DO_2 = \sqrt{8}.$$

Finally then, we have

$$AM = \frac{\sqrt{7}}{\sqrt{8}},$$

and

$$AC = 2AM = \sqrt{\frac{7}{2}}.$$

From Various Kürschák Competitions

1. From the 1982 Competition

(An alternative solution is given in *Crux Mathematicorum*, 1989, 228.)

Suppose each of the integers is colored one of 100 different colors in such a way that

(i) each of the 100 colors is actually used, and
(ii) whenever two intervals $[a, b]$ and $[c, d]$ with integral endpoints have the *same length*,
both begin on the left with the same color (i.e., on a and c), and
both end on the right with the same color (i.e., on b and d, whether or not this is the same color as on a and c), then *the entire intervals are colored in identical fashion*, that is, for each x in the range $0 \leq x \leq b - a$, the pair of integers $a + x$ and $c + x$ are the same color.

FIGURE 51

Prove that the integers -1982 and $+1982$ must have different colors.

Solution

Part 1

Let a pair of integers having the same color be called a **monochromatic** pair and the distance between them a monochromatic distance. Consider two monochromatic pairs (a, b) and (c, d) of colors i and j, respectively, i and j the same or different, which have the same monochromatic distance s (Figure 52). In this case, the intervals $[a, c]$ and $[b, d]$ have the same length and satisfy condition (ii) given in the problem, implying that they are colored identically throughout. Therefore the integers

$a+1$ and $b+1$ must be the same color (k),
$a+2$ and $b+2$ must be the same color (m),
....
b and $b+s$ must be the same color (i again),
$b+1$ and $b+s+1$ must be the same color (k again), and so on.

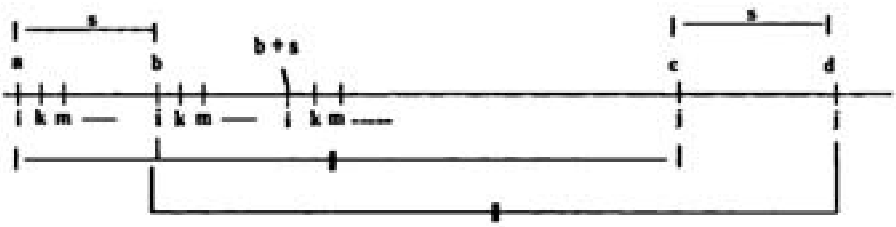


FIGURE 52

Thus $[a, c]$ is colored by abutting blocks, each colored as the interval $[a, b]$, which has length s. (If the length of $[a, c]$ is $q \cdot s + r$, where $0 \leq r < s$, the block will occur q times in $[a, c]$, followed by the first r integers as a part block at the end.)

Part 2

Since there are only 100 colors, the pigeonhole principle implies that every interval of 101 integers must contain two integers of the same color. Since 101 consecutive integers only occupy an interval of length 100, each interval of length 100 gives rise to at least one monochromatic distance d in the bounded range $\{1, 2, \ldots, 100\}$. Accordingly, any *infinite* interval on the number line, containing an infinity of abutting subintervals of length 100, gives rise to an

infinity of such bounded monochromatic distances, and consequently at least one of the distances d must occur infinitely often over the interval.

Part 3

Since each of the 100 colors is actually used, let an integer of each color be chosen, and let $[p, q]$ be any finite interval which contains all these integers. Let $[x, y]$ be an interval that contains both $[p, q]$ and the interval $[-1982, +1982]$ (the latter may already be in $[p, q]$), and containing a buffer of 100 or so on each side of these intervals (Figure 53).

FIGURE 53

By section 2 above, the infinite interval $[y, \infty)$ gives rise to a monochromatic distance d_1 that occurs infinitely often, and similarly the interval $(-\infty, x]$ has an infinitely occuring monochromatic distance d_2, where each of d_1 and d_2 lies in the range $\{1, 2, \ldots, 100\}$.

Now, let (a, b) and (c, d) be two monochromatic pairs with distance d_1 in $[y, \infty)$. Since there are infinitely many such intervals in $[y, \infty]$, we can choose two that are separated by a distance that is as large as we like. As shown in section 1 above, the integers between $[a, b]$ and $[c, d]$ are colored by repeating the block of colors in the interval $[a, b]$. Thus, by taking $[a, b]$ and $[c, d]$ appropriately far apart and by combining such adjacent blocks of length d_1, we see that there exist monochromatic distances in the infinite interval $[y, \infty)$ which are equal to every finite multiple kd_1 of d_1 (Figure 54). Similarly, in $(-\infty, x]$ there are monochromatic distances of all finite multiples of its infinitely occurring distance d_2.

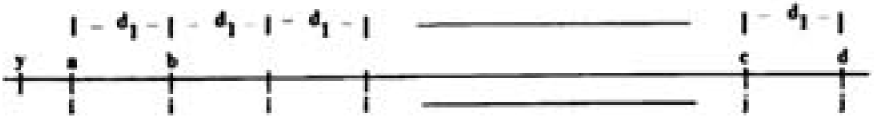


FIGURE 54

It follows, then, that there is such a monochromatic multiple $d_2 \cdot d_1$ in $[y, \infty)$ and an equal monochromatic multiple $d_1 \cdot d_2$ in $(-\infty, x]$, and we conclude that

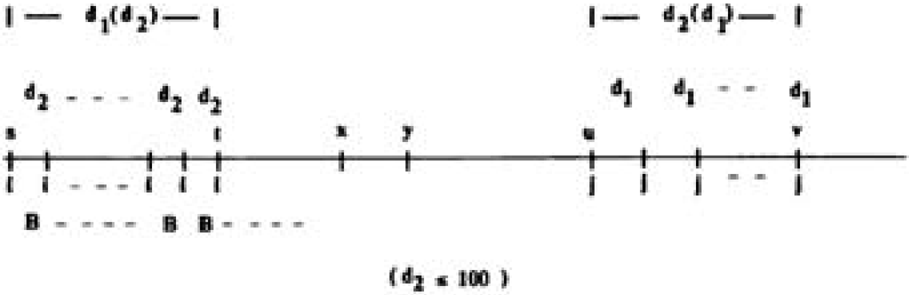


FIGURE 55

there exist two monochromatic pairs (s, t) and (u, v), each with monochromatic distance $d_1 d_2$, one on each side of the interval $[x, y]$ (Figure 55).

Part 4

The intervals $[s, u]$ and $[t, v]$ are therefore the same length and, by condition (ii), they are identically colored. As we saw in section 1 above, the integers starting at s are colored by repeating the colors in the block $[s, t]$ at abutting intervals. But $[s, t]$ itself is colored simply by abutting d_1 copies of a block B of length d_2. Therefore throughout $[s, u]$, the coloring is obtained by abutting copies of this block B.

Since s and u straddle $[x, y]$, the entire interval $[x, y]$ is colored by abutting copies of B, and so the only colors that occur in $[x, y]$ come from B. But $[x, y]$ contains $[p, q]$, which itself contains all 100 colors. Thus B must contain all the colors, and since its length $d_2 \leq 100$, it follows that $d_2 = 100$ and that B is some fixed arrangement of all 100 of the different colors.

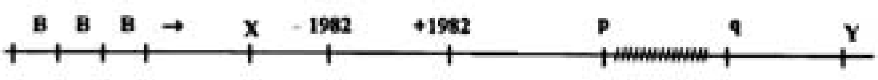


FIGURE 56

Since $[x, y]$, which includes the interval $[-1982, +1982]$ and a buffer of at least 100, is colored by these abutting copies of B, the color on -1982, occurring at intervals of 100, will be found again on the integers

$$-1882, -1782, \ldots, -82, 18, 118, \ldots, 1918, 2018,$$

but not on any integers in between these values, in particular not on the integer $+1982$.

2. From the 1983 Competition

The n points $P_1, P_2, \ldots, P_n$ and also a point Q are given in the plane. If these points are situated so that for each pair (P_i, P_j) there is a third given point P_k that completes a triangle $P_iP_jP_k$ containing the point Q in its interior, prove that n *must be odd.*

Solution

Let the n rays $QP_1, QP_2, \ldots, QP_n$ be colored blue and their images under the half-turn about Q be colored red (the images are dotted in Figure 57). Clearly the n blue rays pair up with their respective red images to form n straight lines.

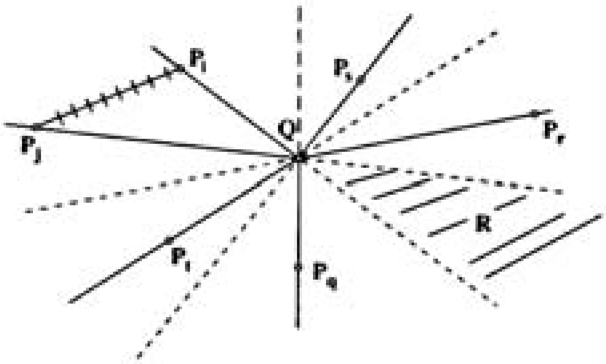


FIGURE 57

Now, in the fan of rays about Q, suppose some two blue rays QP_i and QP_j were to be *consecutive*. In this case, a third point P_k that completes a triangle $P_iP_jP_k$, containing Q in its interior, would have to lie in the opposite region R between the red images of QP_i and QP_j. But, since these red images are also consecutive in the fan, there is no blue ray in between them and therefore, contrary to the given condition, there could be no point P_k to go with the pair (P_i, P_j). Thus it follows that no two blue rays in the fan can be consecutive.

Now let's remove all the lines from our figure and proceed to put them back one at a time in any order, keeping track at each stage of the number b of *pairs of consecutive blue rays* in the growing fan. When the number of lines is $l = 2$, it is clear that b must be 1 (Figure 58). Beyond this stage, there are only three possible ways in which a new line can enter the configuration:

the *blue* ray of the new line must occcur in the fan either

(i) between two consecutive blue rays,
(ii) between a blue ray and a red one, or
(iii) between two consecutive red rays.

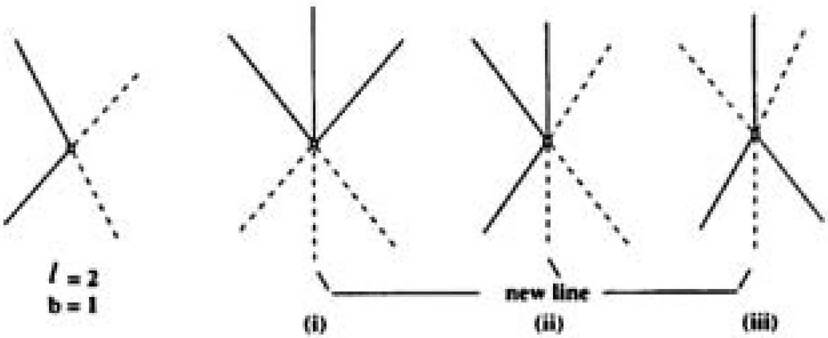


FIGURE 58

From the figures it follows easily that

in case (i), b increases to $b + 1$ since two consecutive blue pairs are created while the newly-partitioned one is destroyed;

in case (ii), b again increases to $b + 1$ with the creation of a single new consecutive blue pair;

in case (iii), b drops to $b - 1$ since an existing blue pair is destroyed by the new line's red ray.

At every stage, then, the *parity* of b changes. Since the number of lines also changes from l to $l + 1$, at each stage both l and b change parity. Thus the values of l and b are either of the same parity throughout the entire procedure or they are of opposite parity throughout. Since $b = 1$ when $l = 2$, it must be that they are always of opposite parity.

Accordingly, at the end of our restoration procedure, the numbers n and b must be of opposite parity. But we have seen that, at this stage, there are no pairs of consecutive blue rays in the fan. For the given configuration, then, $b = 0$, an even number, and therefore n must be odd.

We conclude this section with the entire slate of three problems from the 1985 competition.

3. Problem 1

(An alternative solution is given in 1991, 134)

An arbitrary convex $(n+1)$-gon $P = P_0P_1 \cdots P_n$ is partitioned into $n-1$ triangles by drawing $n-2$ non-intersecting diagonals. Prove that the triangles can be numbered $1, 2, \ldots, n-1$ in such a way that P_i is a vertex of triangle i for all $i = 1, 2, \ldots, n-1$.

Solution

Because there are only two ways of triangulating a quadrilateral, it is evident from the figures that the required numbering can always be accomplished for $n = 3$.

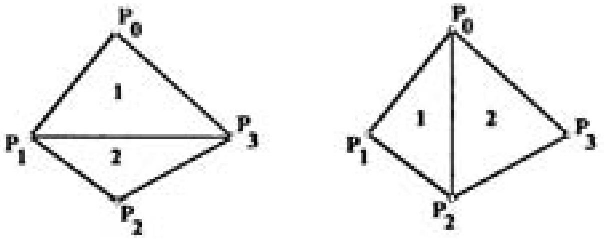


FIGURE 59

Since the nature of the problem seems to invite an attempt by the method of induction, let's assume that the required numbering is always possible for convex polygons $P_0P_1 \cdots P_{n-1}$, where $n - 1 \geq 3$, and turn our attention to a convex $(n+1)$-gon $P = P_0P_1 \cdots P_n$. In general, there are many ways in which our polygon can be triangulated by non-intersecting diagonals, and we must show that the required numbering can be accomplished in every case (for an $(n+1)$-gon, the number of different triangulations is the $(n-1)$th Catalan number $c_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$; a nice proof is given in my Mathematical Gems I, 130–134).

In order to apply the induction hypothesis, we need some way of reducing the given $(n+1)$-gon P to a convex n-gon Q. It would be nice if some diagonal were to cut off a single vertex P_i from the polygon; in this case, a suitable reduction would be obtained simply by deleting the triangle $P_{i-1}P_iP_{i+1}$. The diagonal $P_{i-1}P_{i+1}$ would thus become a side of the new polygon Q and generally, in order to apply the induction hypothesis, we would need, as we go

around Q, to relabel the vertices $P_{i+1}, P_{i+2}, \ldots, P_n$, lying beyond P_{i-1}, with the names $P_i, P_{i+1}, \ldots, P_{n-1}$, respectively.

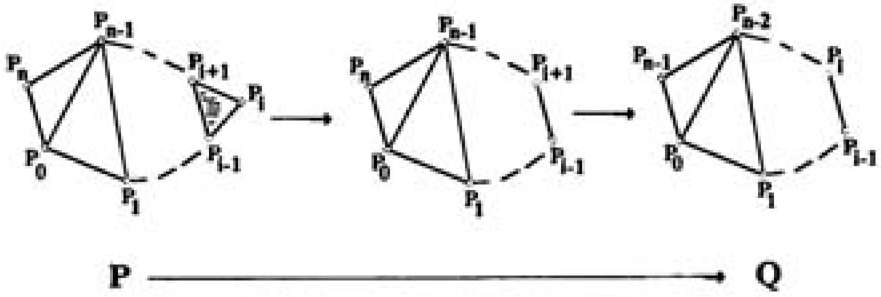


FIGURE 60

This renaming procedure needs to be adjusted if the deleted vertex happens to be either P_0 or P_n. As we shall see, however, these contingencies can always be avoided, for every triangulation has at least one diagonal which cuts off a single vertex which is neither P_0 nor P_n.

Clearly, each side of the polygon P belongs to some triangle in the decomposition and while it is possible for all three sides of a triangle to be diagonals, no triangle can contain more than two sides of P (recall $n - 1 \geq 3$, implying that P itself is not a triangle). Therefore, the $n + 1$ sides of P must be distributed over $n - 1$ or fewer triangles, and so, on at least two occasions, a triangle must contain two sides of P. Since two such sides would have to be adjacent in P, there are at least two diagonals that each cut off a single vertex from P; and since the diagonals are non-intersecting, no two vertices thus cut off can be adjacent in P. Hence at least one of the two or more vertices cut off must fail to belong to the adjacent pair (P_0, P_n). It follows, then, that no matter which of the many triangulations might be used, it is always possible to reduce our given $(n + 1)$-gon P to a convex n-gon $Q = P_0P_1 \cdots P_{n-1}$ in which the vertices $P_{i+1}, \ldots, P_n$,

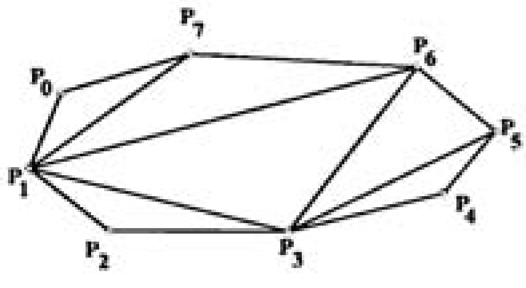


FIGURE 61

beyond the deleted vertex P_i, which is neither P_0 nor P_n, have been respectively renamed $P_i, \ldots, P_{n-1}$.

Using the induction hypothesis, let the triangles in Q be numbered so that each of the vertices $P_1, P_2, \ldots, P_{n-2}$ in Q is a vertex, respectively, of triangle $1, 2, \ldots, n-2$. Since the deleted vertex P_i is neither P_0 nor P_n, i must be one of the numbers $1, 2, \ldots, n-1$. Now let the numbering of the triangles in Q be transferred to P, and let the deleted triangle $P_{i-1}P_iP_{i+1}$ be assigned the number i. If $i = n-1$, numbering this restored triangle with $n-1$ happily completes the numbering of the triangles from 1 through $n-1$, as desired. Otherwise $i \leq n-2$, and we are saddled with two triangles numbered i and no triangle numbered $n-1$.

The remedy, of course, is to renumber the triangles $i, i+1, \ldots, n-2$ of Q with the numbers $i+1, i+2, \ldots, n-1$ respectively; then Q directly numbers triangles $1, 2, \ldots, i-1$, the delted triangle is number i, and the remaining triangles in Q, each with its number increased by 1, label triangles $i+1, i+2, \ldots, n-1$. Recall that, for $k \in \{i+1, i+2, \ldots, n\}$, the vertex P_k in P was renamed P_{k-1} in the formation of Q. Thus, while the triangles are numbered in Q so that P_{k-1} is a vertex of triangle $k-1$, when the numbering is transferred to P, it is P_k that is associated with triangle $k-1$ (for $k \geq i+1$). Hence the elevation of the names of these triangles from $k-1$ to k restores the required association of P_k with triangle k.

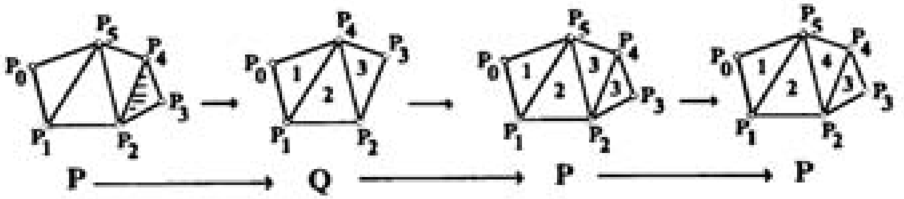


FIGURE 62

By induction, then, we conclude that the required numbering is always possible.

4. Problem 2

Let n be a positive integer. For each prime divisor p of n, consider the greatest power of p that does *not exceed* n. Let the sum of all these greatest prime powers be called the power-sum of n, and be denoted by $S(n)$. Prove that there exist infinitely many positive integers n which are less than their respective power-sums, i.e., for which $n < S(n)$.

Solution

One never knows how difficult an innocent-sounding problem in elementary number theory might be and a Kürschák problem of this kind is always a large question-mark. However, the problem at hand has an exceptionally quick and easy solution, for $n = 2^{k+1} + 2$ satisfies $S(n) > n$ for every positive integer k.

Since

$$n = 2^{k+1} + 2 = 2(2^k + 1),$$

containing an odd factor $2^k + 1$, which is at least 3, n must have some odd prime divisor q which is also at least 3. Hence

$$S(2^{k+1} + 2) \geq 2^{k+1} + q^1 \geq 2^{k+1} + 3 > 2^{k+1} + 2,$$

and the conclusion follows already.

It is almost as easy to see that $n = 2p$, p an odd prime, also satisfies $S(n) > n$. The greatest power 2^t which does not exceed an even number n is always more than half the number:

if, on the contrary, $2^t \leq \frac{n}{2}$, then $2^{t+1} \leq n$,
contradicting the maximum character of 2^t.

For $n = 2p$, then, the greatest power 2^t, which does not exceed n, must be greater than p:

$$2^t > p.$$

Since p is an odd prime, then $2p < pp = p^2$, implying that the greatest power of p which does not exceed n is p itself. Hence

$$S(n) = S(2p) = 2^t + p > p + p = 2p = n, \text{ as claimed.}$$

Comments

Clearly $n = p$, a prime, yields $S(n) = n$, and thus there is an infinity of positive integers which satisfy each of $S(n) = n$ and $S(n) > n$. If the following is any indication, it seems to require a greater effort to show that there is also an infinity of solutions of $S(n) < n$.

By Bertrand's Postulate, there exists a prime number between n and $2n$, for every positive integer $n \geq 2$. Let p be any odd prime and q a prime between p and $2p$. As we shall see, it follows that $S(pq) < pq$.

From $p < q < 2p$, we have $p^2 < pq < 2p^2 < p^3$, since $p \geq 3$, showing that the greatest power of p that does not exceed pq is p^2. Also, $pq < qq = q^2$, implying that the greatest power of q that does not exceed n is q itself. Therefore

$$S(pq) = p^2 + q.$$

Now, q must also be an odd prime, and so the difference $q - p$ must be at least 2. Thus we have

$$q - p \geq 2, \quad \text{giving} \quad p(q - p) \geq 2p > q,$$
$$pq - p^2 > q, \quad -pq + p^2 < -q, \quad \text{and} \quad p^2 + q < pq,$$

that is, $S(pq) < pq$, as claimed.

5. Problem 3

(An alternative solution is given in 1990, 6)

Suppose each vertex of a triangle ABC is reflected in the opposite side of the triangle to give the three image points X, Y, and Z. Prove that the area of triangle XYZ is never as much as five times the area of the original triangle ABC.

Solution

Let the area of triangle ABC be denoted by Δ. Since the reflections copy triangle ABC in the image-triangles XBC, YAC, and ZAB, these triangles also have

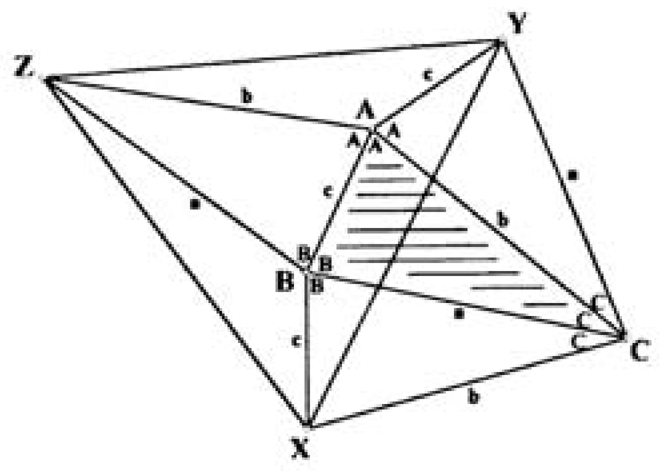


FIGURE 63

area Δ. Now, the area of triangle XYZ is given in terms of these copies of ABC and the three "outer" triangles CXY, AYZ, and BZX as follows (Figure 63):

$$\begin{aligned}\Delta XYZ &= \Delta ABC + \Delta XBC + \Delta YAC + \Delta ZAB \\ &\quad \pm \Delta AYZ \pm \Delta BZX \pm \Delta CXY \\ &= 4\Delta \pm \Delta AYZ \pm \Delta BZX \pm \Delta CXY,\end{aligned}$$

where each of the last three triangles may need to be added or subtracted as the shape of triangle ABC dictates. In the case shown, the first two should be added and the third subtracted.

Since each angle θ of triangle ABC occurs three times at the vertex of the same name, we can see from Figure 63 that the adjacent outer triangle should be subtracted when $3\theta < 180°$ (this forces the triangle to lie outside ΔXYZ), added when $3\theta > 180°$ (putting the triangle inside ΔXYZ), and that it collapses harmlessly to zero area when $3\theta = 180°$. Recalling the formula $\frac{1}{2}pq \sin R$ for the area of ΔPQR, and that $\sin(360° - 3\theta) = -\sin 3\theta$, in the case shown we have

$$\begin{aligned}\Delta XYZ &= 4\Delta + \Delta AYZ + \Delta BZX - \Delta CXY \\ &= 4\Delta + \frac{1}{2}bc\sin(360° - 3A) + \frac{1}{2}ac\sin(360° - 3B) - \frac{1}{2}ab\sin 3C \\ &= 4\Delta - \frac{1}{2}bc\sin 3A - \frac{1}{2}ac\sin 3B - \frac{1}{2}ab\sin 3C.\end{aligned}$$

As a matter of fact, this expression gives the correct area of ΔXYZ in *all* cases:

> when $3\theta > 180°$, in which case the corresponding outer triangle should be added, $\sin 3\theta$ is negative, making $-\sin 3\theta$ appropriately positive,
>
> and
>
> when $3\theta < 180°$, calling for the subtraction of the triangle, $-\sin 3\theta$ is obligingly negative.

Therefore, in order to show that triangle XYZ never amounts to as much as 5Δ, we need to show that

$$-\frac{1}{2}bc\sin 3A - \frac{1}{2}ac\sin 3B - \frac{1}{2}ab\sin 3C < \Delta,$$

or any of the following equivalent forms:

$$\frac{1}{2}bc\sin 3A + \frac{1}{2}ac\sin 3B + \frac{1}{2}ab\sin 3C > -\Delta,$$

$$\frac{1}{2}bc(3\sin A - 4\sin^3 A) + \frac{1}{2}ac(3\sin B - 4\sin^3 B)$$
$$+ \frac{1}{2}ab(3\sin C - 4\sin^3 C) > -\Delta,$$
$$3\Delta - 4\Delta\sin^2 A + 3\Delta - 4\Delta\sin^2 B + 3\Delta - 4\Delta\sin^2 C > -\Delta,$$
$$10\Delta > 4\Delta(\sin^2 A + \sin^2 B + \sin^2 C),$$
$$\frac{5}{2} > \sin^2 A + \sin^2 B + \sin^2 C,$$

an interesting challenge.

As a first step in establishing this last relation, let us show that

$$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C.$$

Clearly,

$$\begin{aligned}\sin^2 A + \sin^2 B + \sin^2 C &= 1 - \cos^2 A + 1 - \cos^2 B + \sin^2 C\\ &= 2 + (\sin^2 C - \cos^2 A - \cos^2 B).\end{aligned}$$

To complete this first step, then, we need to show that

$$2\cos A\cos B\cos C = \sin^2 C - \cos^2 A - \cos^2 B.$$

Noting that

$$2\cos\theta\cos\phi = \cos(\theta+\phi) + \cos(\theta-\phi), \quad \text{and} \quad A + B + C = 180^\circ,$$

we have

$$\begin{aligned}2\cos A\cos B\cos C &= [\cos(A+B) + \cos(A-B)]\cos C\\ &= \cos(180^\circ - C)\cos C + \cos(A-B)\cos C\\ &= -\cos^2 C + \cos(A-B)\cos[180^\circ - (A+B)]\\ &= -\cos^2 C - \cos(A-B)\cos(A+B)\\ &= -\cos^2 C - \frac{1}{2}[2\cos(A+B)\cos(A-B)]\\ &= -\cos^2 C - \frac{1}{2}(\cos 2A + \cos 2B)\\ &= -\cos^2 C - \frac{1}{2}(2\cos^2 A - 1 + 2\cos^2 B - 1)\\ &= -\cos^2 C - \cos^2 A - \cos^2 B + 1\\ &= (1 - \cos^2 C) - \cos^2 A - \cos^2 B\\ &= \sin^2 C - \cos^2 A - \cos^2 B, \quad \text{as desired.}\end{aligned}$$

To finish the task we need to show that

$$2 + 2\cos A \cos B \cos C < \frac{5}{2},$$

i.e., that

$$2\cos A \cos B \cos C < \frac{1}{2}.$$

As noted above,

$$2\cos A \cos B \cos C = [\cos(A+B) + \cos(A-B)]\cos C.$$

Now, A, B, and C are some positive angles that add up to 180°. Suppose some two of them, say A and B, were to be unequal. In this case, these particular values of A, B, and C could not provide the expression

$$[\cos(A+B) + \cos(A-B)]\cos C$$

with the greatest value it is capable of assuming, for the adjusted set of values (A', B', C'), given by

$$A' = B' = \frac{A+B}{2}, \quad C' = C,$$

keep $\cos(A'+B')$ and $\cos C'$, respectively, at the same values as $\cos(A+B)$ and $\cos C$, but *increase* the remaining term to $\cos(A'-B') = \cos 0 = 1$ (in the expression at hand, $\cos(A-B)$ is less than 1 since A and B are unequal). From the symmetry of $2\cos A \cos B \cos C$, it follows that its maximum value is similarly not attained whenever *any* two of A, B, and C are unequal. Since its continuity assures us that the function does have a maximum value over the interval $0 < A, B, C < 180^\circ$, we conclude that

$$\max(2\cos A \cos B \cos C) = 2\cos 60^\circ \cos 60^\circ \cos 60^\circ = 2\left(\frac{1}{2}\right)^3 = \frac{1}{4},$$

which is well below the proposed ceiling of $\frac{1}{2}$, and our argument is complete.

Two Questions from the 1986 National Junior High School Mathematics Competition of the People's Republic of China

(*Crux Mathematicorum*, 1988, 130)

Questions at the junior high school level are generally not sufficiently challenging to attract our attention, but the two given below are so appealing, especially the first one, that I can't resist them.

Problem 1

The lengths AB, BC, CD, DA of quadrilateral $ABCD$ are $1, 9, 8, 6$, respectively. Which of the answers given below correctly describes the following set of five statements?

(i) Quadrilateral $ABCD$ can be circumscribed about a circle.
(ii) Quadrilateral $ABCD$ cannot be inscribed in a circle.
(iii) The diagonals AC and BD are not in perpendicular directions.
(iv) Angle $ADC \geq 90°$.
(v) $\triangle BCD$ is isosceles.

A:	(i) is true,	(ii) is false,	(iv) is true.
B:	(iii) is true,	(iv) is false,	(v) is true.
C:	(iii) is true,	(iv) is false,	(v) is false.
D:	(ii) is false,	(iii) is false,	(iv) is true.

Solution

Summarizing the statements and answers in a table is a big help in getting an overall grasp of the situation; and everybody knows what the Chinese say about a picture.

	(i)	(ii)	(iii)	(iv)	(v)
A	T	F		T	
B			T	F	T
C			T	F	F
D		F	F	T	

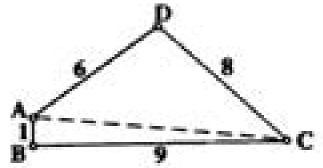


FIGURE 64

From the table it is clear that statement (iv) figures most prominently in the answers. Therefore let us begin by having a look at (iv), which asserts $\measuredangle ADC \geq 90°$. In the event that $\measuredangle ADC = 90°$, then $\triangle ADC$ would be a 6–8–10 right-angled triangle, with hypotenuse $AC = 10$. Now, AC certainly can't be any bigger than 10, for the triangle inequality gives

$$AC \leq AB + BC = 1 + 9.$$

Therefore $\measuredangle ADC > 90°$ is out of the question, and if (iv) holds, then $\measuredangle ADC$ is exactly 90°, which would straighten out ABC and cause the quadrilateral to degenerate into a triangle with B on the side AC (Figure 65).

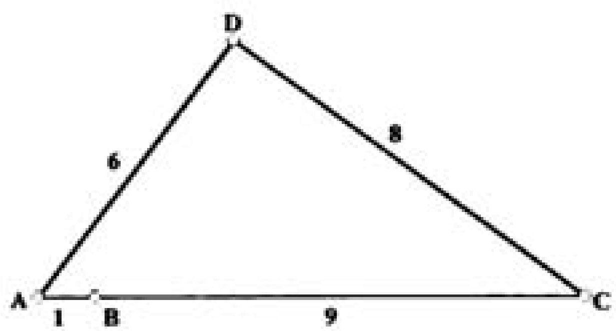


FIGURE 65

As a result, the circle through A, C, and D must fail to go through B, implying that statement (ii) is correct in asserting $ABCD$ is not cyclic. That is to say, (iv) implies (ii), and therefore answers A and D cannot be correct. Of the remaining two answers, one claims (v) is true and the other that it is false. Since this is the only distinction between them, let's see if (v) can help us choose between them.

Statement (v) asserts that $\triangle BCD$ is isosceles. But

$$BD \le BA + AD = 1 + 6 = 7,$$

and therefore, with the other sides of the triangle having lengths 8 and 9, it is clear that (v) is false under all conditions. The correct answer, then, must be C.

Problem 2

Let n be a positive integer and

$$I_n = (n+1)^2 + n - \left[\sqrt{(n+1)^2 + n + 1}\right]^2,$$

where the square brackets denote the greatest integer function.
Then

A: $I_n > 0$ for all n. B: $I_n < 0$ for all n.

C: $I_n = 0$ for all n. D: I_n ranges over positive, negative, and zero values.

Solution

Taking a look at a few initial values of I_n, we get

$$I_1 = 4 + 1 - [\sqrt{6}]^2 = 5 - 4 = 1;$$
$$I_2 = 9 + 2 - [\sqrt{12}]^2 = 11 - 9 = 2;$$
$$I_3 = 16 + 3 - [\sqrt{20}]^2 = 19 - 16 = 3,$$

suggesting the conjecture

$$I_n = n.$$

Clearly,

$$(n+1)^2 < (n+1)(n+2) < (n+2)^2,$$

giving

$$n + 1 < \sqrt{(n+1)(n+2)} < n + 2,$$

making

$$[\sqrt{(n+1)(n+2)}] = n + 1,$$

i.e.,

$$\left[\sqrt{(n+1)[(n+1)+1]}\right] = \left[\sqrt{(n+1)^2+n+1}\right] = n+1.$$

Thus

$$\left[\sqrt{(n+1)^2+n+1}\right]^2 = (n+1)^2,$$

and

$$I_n = (n+1)^2 + n - (n+1)^2 = n, \quad \text{as suspected.}$$

Hence A: $I_n > 0$ for all n is the correct answer.

From the 1986 Spanish Olympiad

(*Crux Mathematicorum*, 1988, 68)

Suppose the lengths of the sides of right-triangle ABC are integers. Let the segments which join the centroid G to the vertices partition the triangle into three smaller triangles with areas x, y, and z. Prove that each of x, y, and z is an *even integer*.

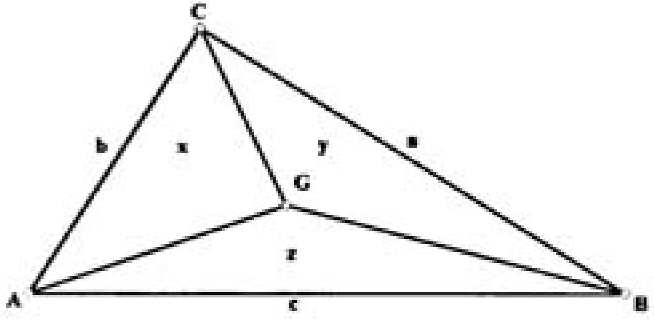


FIGURE 66

Solution

(A similar solution is given in *Crux Mathematicorum*, 1990, 17.) The first thing to notice is that these little triangles are all the same size, namely $\frac{1}{3}\triangle ABC$: for example, median CC' bisects $\triangle ABC$, and since the centroid trisects a median, we have

$$x = \frac{2}{3}\triangle CAC' = \frac{2}{3}\cdot\frac{1}{2}\triangle ABC = \frac{1}{3}\triangle ABC.$$

Similarly for y and z. Hence we need only show that x is an even integer.

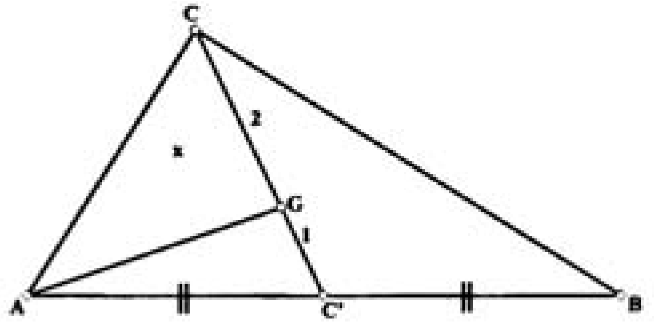


FIGURE 67

Now, it is fairly generally known, and I presume well known to olympiad contestants, that the integers a, b, c, in a Pythagorean triple can be expressed in terms of three positive integers k, m, and n as follows:

$$a = 2kmn, \quad b = k(m^2 - n^2), \quad c = k(m^2 + n^2).$$

In these terms, recalling $\measuredangle C$ is a right angle

$$x = \frac{1}{3}\triangle ABC = \frac{1}{3}\cdot\frac{1}{2}ab = \frac{k^2mn(m^2 - n^2)}{3}.$$

We need to show, then, that not only does 3 divide this numerator N but that 2 does also (in order to make x even). But these things are easily accomplished.

(i) If either m or n is divisible by 3, then so is N; otherwise, $m, n \equiv \pm 1 \pmod 3$, giving $m^2, n^2 \equiv 1 \pmod 3$, making $m^2 - n^2 \equiv 0 \pmod 3$, showing that N is unavoidably divisible by 3.

(ii) If either m or n is divisible by 2, so is N; otherwise, m and n are both odd, making $m^2 - n^2$ even, and the argument is complete.

A Geometric Construction

(P1188, *Crux Mathematicorum*, 1988, 32)

Here is a lovely little geometric construction which has a perfectly simple straightforward solution if you can only think of it.

In the plane of a given circle K, with center O and radius r, points A and B are specified. Construct a chord PQ of K to pass through B and subtend a right angle at A.

This problem comes from Dan Sokolowsky (Williamsburg, Virginia), and the solution is due to George Tsintsifas (Thessaloniki, Greece).

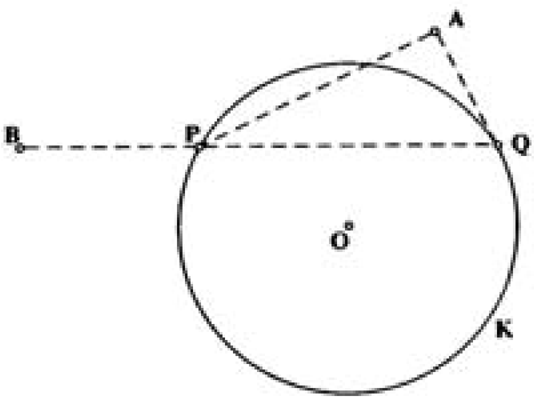


FIGURE 68

Solution

Clearly, if any new point M on the required chord could be found, BM would solve the problem. The question is what point on PQ might be worthwhile trying to locate? Of course, a special point on any chord of a circle is its midpoint M, for the segment MO to the center is perpendicular to the chord. In the case at hand, this seems to be an especially good choice in view of the additional fact that the midpoint of the hypotenuse of a right-triangle is equidistant from the three vertices. It would appear, then, to be worth trying to find two loci that intersect at the midpoint M of PQ.

Since angle BMO is a right angle, we have immediately that the circle on BO as diameter passes through M. Let's hope that other simple relations in the figure will suggest a second locus that goes through M.

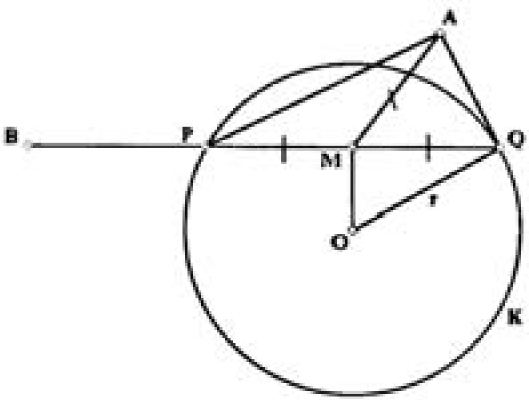


FIGURE 69

In right-triangle OMQ, the Pythagorean theorem gives

$$r^2 = OM^2 + MQ^2,$$

and since $MQ = MA$, we have

$$r^2 = OM^2 + MA^2.$$

Surprisingly, we have already reached the crucial relation, for it remains only to think of the quite well-known theorem that *the sum of the squares of two sides of a triangle is equal to twice the square on the median to the third side plus one-half the square on the third side*:

$$XY^2 + XZ^2 = 2XW^2 + \frac{1}{2}YZ^2.$$

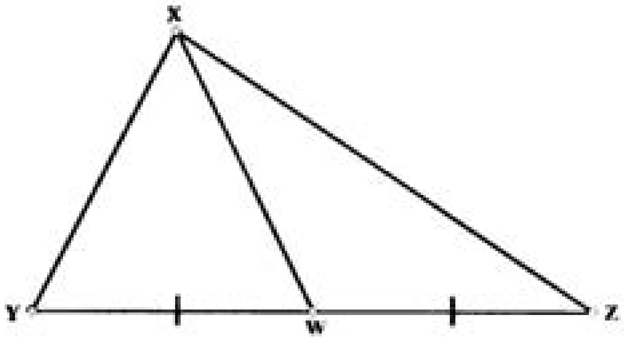


FIGURE 70

(This is an immediate consequence of applying the law of cosines to triangles XYW and XWZ.)

Thus, if MN is the median to OA in triangle OMA (Figure 71), we have

$$r^2 = OM^2 + MA^2 = 2MN^2 + \frac{1}{2}OA^2,$$

from which MN can be determined in terms of the known segments r and OA:

$$MN = \sqrt{\tfrac{1}{2}r^2 - \tfrac{1}{4}OA^2}.$$

Hence the circle with known center N and radius MN is a second locus through M, and the solution is complete.

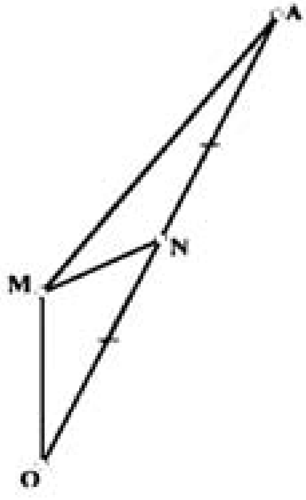


FIGURE 71

An Inequality Involving Logarithms

(P1127, *Crux Mathematicorum*, 1987, 202)

(Proposed by D. S. Mitrinovic, University of Belgrade, Yugoslavia; solved by Murray Klamkin, University of Alberta)

Part (a) (only): If a, b, and c are any real numbers greater than 1, prove, for any exponent $r > 0$, that the sum

$$S = (\log_a bc)^r + (\log_b ca)^r + (\log_c ab)^r \geq 3 \cdot 2^r.$$

Solution

First of all we should note that $\log_x y = \dfrac{\ln y}{\ln x}$ (letting $\log_x y = k$, we have $y = x^k = e^{k \ln x}$, implying $\ln y = k \ln x$). Thus

$$\log_a bc = \frac{\ln bc}{\ln a} = \frac{\ln b + \ln c}{\ln a} = \frac{\ln b}{\ln a} + \frac{\ln c}{\ln a}.$$

Now, by the arithmatic mean–geometric mean inequality, we have

$$\frac{\ln b}{\ln a} + \frac{\ln c}{\ln a} \geq 2\left[\frac{\ln b \cdot \ln c}{(\ln a)^2}\right]^{1/2},$$

and so

$$\log_a bc \geq \frac{2(\ln b \cdot \ln c)^{1/2}}{\ln a},$$

giving

$$(\log_a bc)^r \geq \frac{2^r(\ln b \cdot \ln c)^{r/2}}{(\ln a)^r}.$$

Similarly,

$$(\log_b ca)^r \geq \frac{2^r(\ln c \cdot \ln a)^{r/2}}{(\ln b)^r} \quad \text{and} \quad (\log_c ab)^r \geq \frac{2^r(\ln a \cdot \ln b)^{r/2}}{(\ln c)^r}.$$

Hence the sum S is bounded by

$$S \geq \frac{2^r(\ln b \cdot \ln c)^{r/2}}{(\ln a)^r} + \frac{2^r(\ln c \cdot \ln a)^{r/2}}{(\ln b)^r} + \frac{2^r(\ln a \cdot \ln b)^{r/2}}{(\ln c)^r}.$$

Finally, applying the A. M.–G. M. inequality to the terms on the right side, we have

$$\frac{S}{3} \geq \left[\frac{2^r(\ln b \cdot \ln c)^{r/2}}{(\ln a)^r} \cdot \frac{2^r(\ln c \cdot \ln a)^{r/2}}{(\ln b)^r} \cdot \frac{2^r(\ln a \cdot \ln b)^{r/2}}{(\ln c)^r}\right]^{1/3},$$

and

$$S \geq 3\left[\frac{2^{3r}(\ln a \cdot \ln b \cdot \ln c)^r}{(\ln a \cdot \ln b \cdot \ln c)^r}\right]^{1/3},$$

that is,

$$S \geq 3 \cdot 2^r, \quad \text{as desired.}$$

This proof generalizes directly to the case of n numbers >1:

$$S = \sum_{i=1}^{n} (\log_{a_i} a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n)^r \geq n(n-1)^r.$$

On Isosceles Right-angled Pedal Triangles

(P1239, *Crux Mathematicorum*, 1987, 120)

If perpendiculars are drawn to the sides of triangle ABC from a point P, the triangle formed by the feet D, E, F, is called the *pedal* triangle of P with respect to $\triangle ABC$. The point P can be either inside or outside $\triangle ABC$, or even on a side. In P1239, J. T. Groenman (Arnhem, The Netherlands) posed the following intriguing problem about pedal triangles.

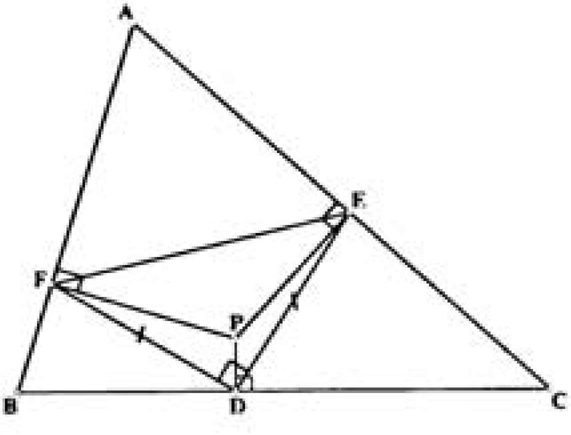


FIGURE 72

Given $\triangle ABC$, where would you pick the point P so that its pedal triangle is an isosceles right-angled triangle?

Solution

We shall not attempt to solve this problem completely, but confine ourselves to points *inside* $\triangle ABC$.

The perpendiculars PD, PE, PF partition the triangle into three quadrilaterals, each of which is clearly *cyclic*. This makes the figure an example of what might be called a "Miquel" configuration. Miquel's theorem establishes the remarkable fact that *if P_1, P_2, and P_3 are chosen arbitrarily on the sides BC, AC, and AB, respectively, of $\triangle ABC$, then the circles AP_2P_3, P_1BP_3, P_1P_2C are always concurrent.*

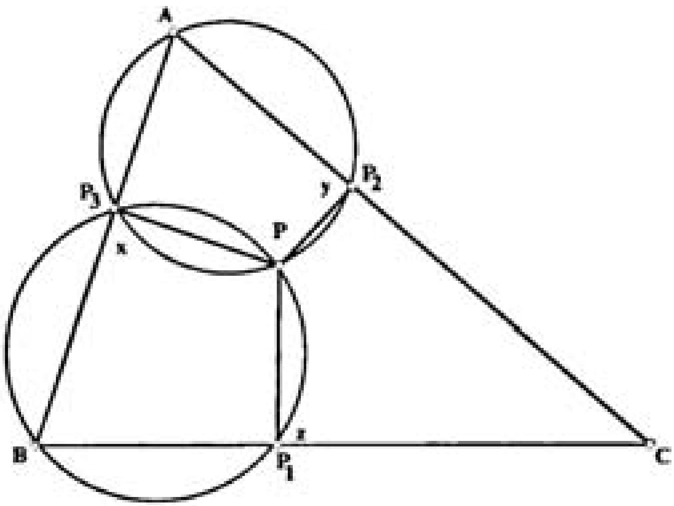


FIGURE 73

The proof is immediate: referring to Figure 73, if P is the intersection of the circles through A and B, then the cyclic quadrilaterals yield $x = y$ and $x = z$, giving $y = z$, implying PP_1CP_2 is also cyclic. P is called the *Miquel point* of $\triangle P_1P_2P_3$, and $\triangle P_1P_2P_3$ is a *Miquel triangle* of P.

Now, the same point P is the Miquel point for an infinite family of Miquel triangles. For any choice of the point P, it is clear from the figure that it is only necessary for the radial lines PP_1, PP_2, PP_3 to meet their respective sides at the *same angle*. The pedal triangle is just the Miquel triangle that results when these angles are right angles.

Now, it so happens that a Miquel configuration always enjoys the following engaging property:

$$\measuredangle BPC = \measuredangle P_3P_1P_2 + A,$$

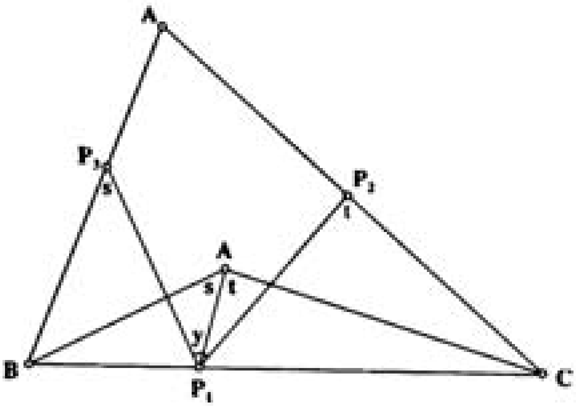


FIGURE 74

i.e., in Figure 74,

$$s + t = y + A.$$

This has a very elementary straightforward proof which I shall pass over in the hope that the following approach might also be of interest.

In the cyclic quadrilaterals, we clearly have

$$\measuredangle BP_3P_1 = s \quad \text{and} \quad \measuredangle P_1P_2C = t.$$

Now, let AB be rotated about P_3 through the angle s to bring it on top of P_3P_1. Next let it be further rotated about P_1 through the angle $-y$ to carry it onto P_1P_2, and finally, let it be rotated about P_2 through angle t to bring it into line with AC. The net effect of all this is to advance our rotating line from the direction of AB to that of AC, i.e., through an angle equal to A. It follows, then, that

$$s - y + t = A, \quad \text{and} \quad s + t = y + A, \quad \text{as claimed.}$$

Thus, in the case of an isosceles right-angled pedal triangle with the vertex of its right angle on BC, we should like angle y to be a right angle. This would require angle BPC to be $90° + A$. That is to say, P must lie somewhere on the arc K of a circle on the chord BC which contains the angle $90° + A$. But this is easily drawn. To get its center T, it is only necessary to construct angles at B and C equal to A (Figure 75(a)), as shown.

In order to make DEF isosceles as well, its other angles must be $45°$, which is similarly accomplished by requiring P to lie on the arc L of a circle on chord AC containing the angle $45° + B$ (in this case, the Miquel angle property gives $\measuredangle CPA = z + B$). This is also easily constructed (Figures 75(b) and (c)).

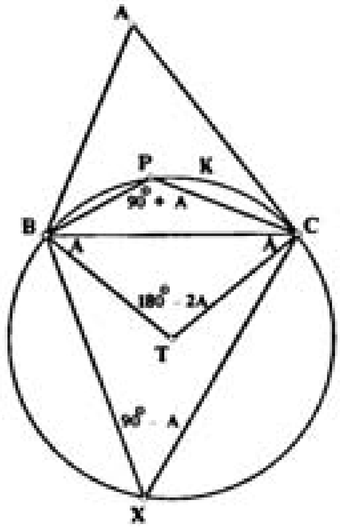


FIGURE 75(a)

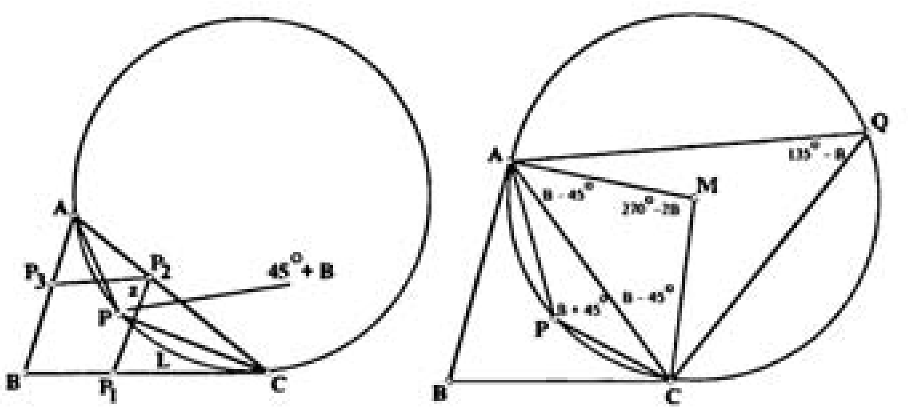


FIGURE 75(b)–(c)

Thus it is really a simple matter to locate P at the intersection of the arcs K and L, and it is clear that P is the *only* point inside $\triangle ABC$ whose pedal triangle DEF is an isosceles right triangle which has the vertex of the right angle at D on BC, i.e., with hypotenuse EF. With the two positions of P corresponding to hypotenuses DF and DE, we have the only three points P *inside* $\triangle ABC$ which solve the problem.

This is as far as we shall pursue the matter, except to note that there are also three other points P *outside* $\triangle ABC$ and that the six points go together in pairs as follows.

If P is the point inside the triangle which puts the right angle at D on BC, then the coresponding point P' outside $\triangle ABC$, which also puts the right angle of its pedal triangle on BC, is found by extending the arc K, described above, into its entire circle, and noting its intersection with the line OP joining P to the circumcenter O of $\triangle ABC$ (Figure 76). (It turns out that P and P' are *inverse* points with respect to the circumcircle of $\triangle ABC$, and this construction yields the inverse P' of P since the circle K is orthogonal to the circumcircle (an easy exercise).)

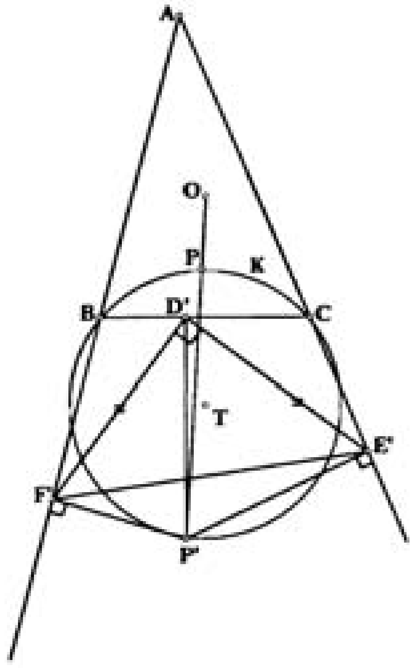


FIGURE 76

Two Problems from the 1987 Austrian Olympiad

(*Crux Mathematicorum*, 1987, 34–35)

Problem 1

Solving the equation

$$x^2 + x - 2 = 0,$$

we easily get

$$(x - 1)(x + 2) = 0,$$

giving the roots 1 and -2. Evidently, then, $x^2 + x - 2$ is a monic polynomial with integer coefficients which has the engaging property that the roots of the corresponding equation are none other than its own final coefficients.

Another example is

$$\begin{aligned} x^3 + x^2 - x - 1 &= 0, \\ x^2(x + 1) - (x + 1) &= 0, \\ (x^2 - 1)(x + 1) &= 0, \end{aligned}$$

with roots $1, -1$, and -1.

In this problem we are required to find all such polynomials: determine all monic polynomials with integer coefficients,

$$P_n(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n,$$

such that the n roots of $P_n(x) = 0$ are its own coefficients $a_1, a_2, \ldots, a_n$.

Solution

This was one of three problems the contestants in this olympiad were given a total of $4\frac{1}{2}$ hours to solve. While it wouldn't surprise me to learn that many of the competitors solved all three of the problems, I don't mind admitting that it took considerably longer than the entire examination period for me to get all the way through this one problem.

Presumably some of the coefficients might be zero. If the total number of zeros among the a_i is $n-k$, the equation $P_n(x) = 0$ would have exactly $n-k$ zero roots and this would imply a common factor of x^{n-k} in $P_n(x)$:

$$\begin{aligned} P_n(x) &= x^n + a_1 x^{n-1} + \cdots + a_k x^{n-k} \\ &= x^{n-k}(x^k + a_1 x^{k-1} + \cdots + a_k). \end{aligned}$$

The final $n-k$ terms of $P_n(x)$ are $0x^{n-k-1} + 0x^{n-k-2} + \cdots + 0$, carrying the $n-k$ zero coefficients corresponding to the zero roots. Since these omitted coefficients account for the full complement of zero coefficients, all the other coefficients, namely $a_1, a_2, \ldots, a_k$, must be nonzero. That is to say, $P_n(x)$ must consist of some power of x times a monic polynomial all of whose integer coefficients are nonzero. The essential part of $P_n(x)$ is this polynomial

$$P_k(x) = x^k + a_1 x^{k-1} + \cdots + a_k, \quad a_i \text{ nonzero,}$$

for, if the roots of such a polynomial are its coefficients a_i, then the same is true of

$$x^t(x^k + a_1 x^{k-1} + \cdots + a_k), \quad \text{for } all\ t = 0, 1, \ldots;$$

although the factor x^t introduces t new zero roots, it also provides t new corresponding zero coefficients at the end. After observing that $P_n(x) = x^n$ is acceptable for all values of n, it is clear that the problem reduces to the determination of all such polynomials $P_k(x)$.

Let's begins with the general observation that $P_k(x)$ always has the informative factorisation

$$\begin{aligned} P_n(x) &= x^k + a_1 x^{k-1} + \cdots + a_k \\ &= (x - a_1)(x - a_2) \cdots (x - a_k). \end{aligned}$$

Equating absolute terms, we get

$$a_k = (-1)^k a_1 a_2 \cdots a_k,$$

and since a_k is nonzero, this gives

$$a_1 a_2 \cdots a_{k-1} = (-1)^k.$$

Because the a's are integers, this implies that each of the coefficients $a_1, a_2, \ldots, a_{k-1}$ must be either $+1$ or -1 as the individual case may be. As a result, the corresponding factors of $P_k(x)$ are either $x - 1$ or $x + 1$ and, letting i denote the number of $+1$'s among $a_1, a_2, \ldots, a_{k-1}$, we have

$$P_k(x) = (x-1)^i (x+1)^{k-1-i}(x - a_k).$$

Now, when these factors are expanded and the multiplication performed, the final term must turn out to be a_k. But, if i were to be an *even* integer, the final term in $(x-1)^i$ would be $+1$, leading to a final term of $-a_k$, since $(x+1)^{k-1-i}$ always ends in $+1$. This would require $a_k = -a_k$, giving $a_k = 0$, a contradiction. Hence *i must be odd.*

In this case, at least one root must be $+1$, and therefore $P_k(1) = 0$. Thus

$$1 + a_1 + a_2 + \cdots + a_k = 0, \quad \text{and} \quad a_1 + a_2 + \cdots + a_k = -1.$$

On the other hand, equating the coefficients of x^{k-1} in the factored form of $P_k(x)$, we see that the sum of the roots is $-a_1$:

$$a_1 + a_2 + \cdots + a_k = -a_1.$$

Let us note in passing, then, the incidental result that a_1 is always 1.

More to the point, we have

$$a_k = -1 - (a_1 + a_2 + \cdots + a_{k-1}),$$

and since $a_1, a_2, \ldots, a_{k-1}$ consist of i $+1$'s and $(k-1-i)$ -1's, the value of

$$a_1 + a_2 + \cdots + a_{k-1} = i - (k-1-i) = -k + 2i + 1,$$

giving

$$a_k = -1 + k - 2i - 1 = k - 2i - 2,$$

and

$$P_k(x) = (x-1)^i (x+1)^{k-1-i}[x - (k - 2i - 2)].$$

Since it is always a good idea to get one's footing by looking at a few initial cases, let's consider the first few values of k.

$k = 1$: This case is pretty trivial, for $P_1(x) = x + a_1 = 0$ requires the mandatory root a_1 to be $-a_1$, forcing $a_1 = 0$, a contradiction. Thus there is no $P_n(x)$ for $k = 1$, and the only $P_n(x)$ for $n = 1$ is $P_1(x) = x$, contributed by the universal $P_n(x) = x^n$.

$k = 2$: In this case, the odd integer i can only be 1 (recall $i < k$), making $a_k = k - 2i - 2 = -2$, and yielding

$$P_2(x) = (x-1)^1 (x+1)^0 (x+2) = x^2 + x - 2,$$

confirming the first example given in the problem.

$k = 3$: Again, the odd integer i can only be 1, and we have $a_k = k - 2i - 2 = -1$, giving

$$P_3(x) = (x-1)^1(x+1)^1(x+1) = x^3 + x^2 - x - 1,$$

confirming the second example given in the problem.

$k = 4$: This time i can presumably be either 1 or 3.

For $i = 1$, however, we get $a_k = k - 2i - 2 = 0$, a contradiction.
For $i = 3$, we have $a_k = -4$, and

$$\begin{aligned} P_4(x) &= (x-1)^3(x+1)^0(x+4) \\ &= (x^3 - 3x^2 + 3x - 1)(x+4) \\ &= x^4 + x^3 - 9x^2 + 11x - 4, \end{aligned}$$

in which $a_2 = -9$, $a_3 = 11$ are not the required $+1$ or -1. Thus there is no $P_k(x)$ for $k = 4$.

k = 5: With $i = 1$, we have

$$\begin{aligned} P_5(x) &= (x-1)^1(x+1)^3(x-1) \\ &= (x^2 - 2x + 1)(x^3 + 3x^2 + 3x + 1) \\ &= x^5 + x^4 - 2x^3 + \cdots, \end{aligned}$$

with the inappropriate $a_2 = -2$.
With $i = 3$,

$$\begin{aligned} P_5(x) &= (x-1)^3(x+1)^1(x+3) \\ &= (x^3 - 3x^2 + 3x - 1)(x^2 + 4x + 3) \\ &= x^5 + x^4 - 6x^3 + \cdots, \end{aligned}$$

with $a_2 = -6$. Thus there is no $P_k(x)$ for $k = 5$, either.

At this point one might wonder whether any more $P_k(x)$ will ever be found. Unfortunately, it is not evident how to proceed confirming this suspicion. Since our limited experience has been that the value of a_2 has been faulty, perhaps we can show that, for larger values of k, a_2 never turns out to be $+1$ or -1. To this end, let's incorporate the odd parity of i into our formula for $P_k(x)$ with the substitution $i = 2j - 1$. Then j is a positive integer, and $a_k = k - 2i - 2 = k - 4j$, and we have

$$P_k(x) = (x-1)^{2j-1}(x+1)^{k-2j}[x - (k - 4j)].$$

Although somewhat costly in elegance, there doesn't seem to be any way around expanding these powers and actually calculating the term in x^{k-2}. Accordingly, we get

$$(x-1)^{2j-1} = x^{2j-1} - (2j-1)x^{2j-2} + \frac{(2j-1)(2j-2)}{2}x^{2j-3} + \cdots,$$

$$(x+1)^{k-2j} = x^{k-2j} + (k-2j)x^{k-2j-1} + \frac{(k-2j)(k-2j-1)}{2}x^{k-2j-2} + \cdots,$$

$$x - a_k = x - (k-4j).$$

The five terms in x^{k-2} provide a coefficient of

$$a_2 = 1\cdot 1\cdot \frac{(2j-1)(2j-2)}{2} + 1\cdot(k-2j)[-(2j-1)] + 1\cdot\frac{(k-2j)(k-2j-1)}{2}\cdot 1$$
$$+ [-(k-4j)]\cdot 1\cdot[-(2j-1)] + [-(k-4j)](k-2j)\cdot 1.$$

For $a_2 = +1$ or -1, then, we would need

$$(2j-1)(j-1) - (k-2j)(2j-1) + \tfrac{1}{2}(k-2j)(k-2j-1)$$
$$+ (k-4j)(2j-1) - (k-4j)(k-2j) = +1 \text{ or } -1,$$
$$2j^2 - 3j + 1 - 2jk + 4j^2 + k - 2j + \tfrac{1}{2}(k^2 - 4jk + 4j^2 - k + 2j)$$
$$+ 2jk - 8j^2 - k + 4j - k^2 + 6jk - 8j^2 = +1 \text{ or } -1,$$
$$-8j^2 + 4jk - \tfrac{1}{2}k - \tfrac{1}{2}k^2 = 0 \text{ or } -2,$$
$$16j^2 - 8jk + k + k^2 = 0 \text{ or } 4,$$

that is, either

$$\text{(i)}\quad 16j^2 - 8jk + (k^2 + k) = 0, \text{ or}$$
$$\text{(ii)}\quad 16j^2 - 8jk + (k^2 + k - 4) = 0.$$

Now, j is a positive integer, and therefore, in solving these quadratics for j, the discriminant D must be nonnegative. In case (i), we have

$$D = 64k^2 - 64(k^2 + k) = -64k < 0,$$

and in case (ii), we get

$$D = 64k^2 - 64(k^2 + k - 4)$$
$$= -64(k-4),$$

which is also negative since $k \geq 5$. These contradictions show that a_2 is never again an acceptable value and that there are indeed no other $P_k(x)$ to be found. Therefore we conclude that $P_n(x)$ is either x, $x^2 + x - 2$, $x^3 + x^2 - x - 1$, or any power of x times one of these:

$$P_n(x) = x^t\cdot x,\ x^t(x^2 + x - 2),\ \text{or}\ x^t(x^3 + x^2 - x - 1), \qquad t = 0, 1, 2, \ldots.$$

Problem 2

(An alternative solution is given in *Crux Mathematicorum*, 1989, 264.)

This problem concerns sequences $x_1x_2\cdots x_n$ in which each x_i is either a, b, or c. Determine the number of these sequences

(i) which have length n,
(ii) begin and end with the letter a, and
(iii) in which adjacent terms are always different letters.

Solution

Even though this kind of problem is a standard topic in elementary combinatorics, it is nonetheless an engaging challenge and it provides an opportunity to demonstrate a powerful and popular combinatorial technique.

In keeping with the best mathematical tradition, let's look briefly at the sequences of a few small values of n. Let the number of sequences of length n be denoted by t_n.

n	Sequences	t_n
1	a	1
2	none	0
3	aba, aca	2
4	$abca, acba$	2
5	$ababa, abaca, acaba,$ $acaca, abcba, acbca$	6

From this short table, one would have to be pretty sharp to spot the general rule that governs the sequence $\{t_n\}$. However, in attempting to see how sequences of length n might be derived from shorter ones, one might notice that a sequence of length n can be obtained by attaching either ba or ca at the end of any sequence of length $n-2$. For example,

$$aba \begin{array}{l} \nearrow ababa \\ \searrow abaca \end{array}, \qquad aca \begin{array}{l} \nearrow acaba \\ \searrow acaca. \end{array}$$

Of course, all the sequences generated in this way have the letter a in the third-last position. Conversely, a sequence of length n whose third-last term is the letter a yields an acceptable sequence of length $n-2$ when its last two terms are

dropped. Thus the number of sequences of length n in which the third-last term is the letter a is $2t_{n-2}$.

For the rest of the sequences of length n, the third-last term is either b or c, and each sequence of this kind provides a single sequence of length $n-1$ by simply deleting its second-last term:

$$a \cdots bca \longrightarrow a \cdots ba,$$
$$a \cdots cba \longrightarrow a \cdots ca.$$

(This is not allowed when the third-last term is the letter a.) Conversely, there is only one possible letter that can be inserted between the last two terms of a sequence, and doing so clearly extends one of length $n-1$ to one of length n. Therefore there are t_{n-1} sequences of length n in which the third-last term is b or c, and we have altogether that

$$t_n = t_{n-1} + 2t_{n-2}.$$

For some competitors the difficulties would be over at this point, for the routine solution of such recursions is covered in many olympiad training programs. However, in case the routine solution of such a recursion is not at your fingertips, let's consider how a neat application of generating functions can lead to a formula for t_n.

If we attach the unkown numbers t_n to the terms of a power series $f(x)$ as follows,

$$f(x) = t_1 + t_2x + t_3x^2 + \cdots + \quad t_nx^{n-1} + \cdots,$$

then

$$x \cdot f(x) = \quad t_1x + t_2x^2 + \cdots + \; t_{n-1}x^{n-1} + \cdots,$$

and

$$2x^2 \cdot f(x) = \qquad 2t_1x^2 + \cdots + 2t_{n-2}x^{n-1} + \cdots.$$

Subtracting the second and third rows from the first, the recursion $t_n - t_{n-1} - 2t_{n-2} = 0$ gives

$$\begin{aligned}(1 - x - 2x^2)f(x) &= t_1 + (t_2 - t_1)x \\ &= 1 - x \quad (\text{recall } t_1 = 1 \text{ and } t_2 = 0),\end{aligned}$$

and therefore

$$f(x) = \frac{1-x}{1-x-2x^2}.$$

Expanding this expression into its partial fractions, we obtain

$$\frac{1-x}{(1-2x)(1+x)}=\frac{A}{1-2x}+\frac{B}{1+x},$$

$$1-x=A(1+x)+B(1-2x).$$

For $x=-1$ and $\frac{1}{2}$, respectively, we find $2=3B$ and $\frac{1}{2}=\frac{3}{2}A$, giving $A=\frac{1}{3}$ and $B=\frac{2}{3}$. Thus

$$\begin{aligned} f(x) &= \frac{1}{3}(1-2x)^{-1}+\frac{2}{3}(1+x)^{-1} \\ &= \frac{1}{3}(1+2x+\cdots+2^{n-1}x^{n-1}+\cdots) \\ &\quad +\frac{2}{3}(1-x+\cdots+(-1)^{n-1}x^{n-1}+\cdots), \end{aligned}$$

from which the coefficient t_n of x^{n-1} is

$$\begin{aligned} t_n &= \frac{1}{3}\cdot 2^{n-1}+\frac{2}{3}\cdot(-1)^{n-1} \\ &= \frac{2^{n-1}+2(-1)^{n-1}}{3}. \end{aligned}$$

From the 1988 Canadian Olympiad (slightly revised)

(*Crux Mathematicorum*, 1988, 163)

Let $S = \{a_1, a_2, \ldots, a_r\}$ be a set of positive integers containing at least two members. For any nonempty subset $A = \{a_i, a_j, \ldots, a_k\}$ of S, let $p(A)$ denote the product of its members: $p(A) = a_i a_j \cdots a_k$. Let $m(S)$ be the average value of these products taken over all nonempty subsets of S.

Among the sets S with $m(S) = 13$, determine one for which there exists a positive integer a_{r+1} whose addition to the set increases m to the value 49:

$$m(S) = m(a_1, a_2, \ldots, a_r) = 13,$$

and

$$m(a_1, a_2, \ldots, a_r, a_{r+1}) = 49.$$

Solution

Since the number of nonempty subsets in a set of size r is $2^r - 1$, we have

$$m(S) = \frac{\sum p(A)}{2^r - 1},$$

from which

$$\begin{aligned}\sum p(A) &= (a_1 + a_2 + \cdots + a_r) + (a_1a_2 + \cdots + a_{r-1}a_r) + \cdots + (a_1a_2 \cdots a_r)\\ &= (2^r - 1)m(S).\end{aligned}$$

Now, in determining the average for the set $\{a_1, a_2, \ldots, a_{r+1}\}$ the new $\sum p(A)$ would be the old $\sum p(A)$ increased by the terms containing the new

integer a_{r+1}. Thus

$$m\{a_1,\ldots,a_{r+1}\} = \frac{1}{2^{r+1}-1}\left[\text{old}\sum p(A) + a_{r+1} + a_{r+1}(a_1+a_2+\cdots+a_r)\right.$$
$$\left. + a_{r+1}(a_1a_2+\cdots+a_{r-1}a_r)+\cdots+a_{r+1}(a_1a_2\ldots a_r)\right]$$
$$= \frac{1}{2^{r+1}-1}[\text{old}\sum p(A) + a_{r+1}(1+\text{old}\sum p(A))]$$
$$= \frac{1}{2^{r+1}-1}[(2^r-1)m(S) + a_{r+1}[1+(2^r-1)m(S)]].$$

Therefore

$$49 = \frac{1}{2^{r+1}-1}[(2^r-1)\cdot 13 + a_{r+1}[1+(2^r-1)\cdot 13]]$$
$$49(2^{r+1}-1) = 13\cdot 2^r - 13 + a_{r+1}(13\cdot 2^r - 12),$$

giving

$$a_{r+1} = \frac{85\cdot 2^r - 36}{13\cdot 2^r - 12}.$$

Now, r has to be a positive integer ≥ 2 which provides an integral value for a_{r+1}. Trying $r = 2$, we get

$$a_3 = \frac{85\cdot 4 - 36}{13\cdot 4 - 12} = \frac{340-36}{52-12} = \frac{304}{40}, \quad \text{which is not an integer;}$$

for $r = 3$, we find

$$a_4 = \frac{85\cdot 8 - 36}{13\cdot 8 - 12} = \frac{680-36}{104-12} = \frac{644}{92} = 7.$$

While we don't know that this is a unique solution, let us pursue the consequences of $r = 3$ and $a_4 = 7$.

With $r = 3$, the value of $2^r - 1$ is 7, and

$$m(S) = \frac{\sum p(A)}{7} = 13,$$

giving

$$\sum p(A) = (a_1+a_2+a_3) + (a_1a_2+a_1a_3+a_2a_3) + (a_1a_2a_3) = 91.$$

We need a solution to this equation in positive integers (a_1, a_2, a_3). Admittedly, this looks like a major problem, but the sight of these symmetric functions can't

help but make us think of the standard expression from the theory of equations

$$(x + a_1)(x + a_2)(x + a_3) \\ = x^3 + (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3)x + a_1a_2a_3.$$

For $x = 1$, then, we have

$$(1 + a_1)(1 + a_2)(1 + a_3) = 1 + 91 = 92.$$

Since the prime decomposition of 92 is $2 \cdot 2 \cdot 23$, the only solution in positive integers is (1, 1, 22).

Thus $r = 3$ has not been a disappointment and we conclude that an acceptable answer is

$$S = \{a_1, a_2, \ldots, a_r\} = \{1, 1, 22\},$$

and

$$\{a_1, a_2, \ldots, a_r, a_{r+1}\} = \{1, 1, 22, 7\}.$$

There is no doubt that these values must give $m(S) = 13$, but I'm sure we will all rest easier after confirming that $m(1, 1, 22, 7) = 49$.

$$\begin{aligned} m(1,1,22,7) &= \frac{1}{2^4 - 1}\big[(1 + 1 + 22 + 7) + (1 + 22 + 7 + 22 + 7 + 154) \\ &\quad + (22 + 7 + 154 + 154) + 154\big] \\ &= \frac{1}{15}(31 + 213 + 337 + 154) \\ &= \frac{735}{15} \\ &= 49. \end{aligned}$$

A Problem on Closed Sets

Suppose each point on the circumference of a circle is colored red or blue, possibly both red and blue, so that each set of colored points is *closed*, i.e., contains all its limit points. Prove that, if the red set R does not determine a chord of every possible length, then the blue set B will be bound to do so.

Solution

Clearly the claim holds trivially if only one color is used on the circle.

Suppose, then, that both colors are actually used and that P is any red point. Now, there are only two cases concerning the occurrence of blue points in the vicinity of P (Figure 77).

(i) *There are blue points that are arbitrarily close to P:*

In this case, P is a limit point of the blue set B, and because B is closed, $P \in B$, making P both red and blue.

(ii) *There exists a blue point Q that is closest to P:*

In this case, the entire open arc PQ must be red, making Q a limit point of R, and since R is closed, then Q must also be a red point. In any case, *some point Z on the circle must be both red and blue.*

Suppose, then, that R were to fail to determine a chord of some length x. Since Z is a red point, a chord ZY of length x cannot be determined except by a blue point Y. Now, let chords ZN and YM of an arbitrary length d be marked off in the same cyclic direction (Figure 78). In this case, the chord MN must also be of length x, and since this length is not achieved by any pair of red points, it follows that at least one of M or N must be blue. As a result, either ZN or MY

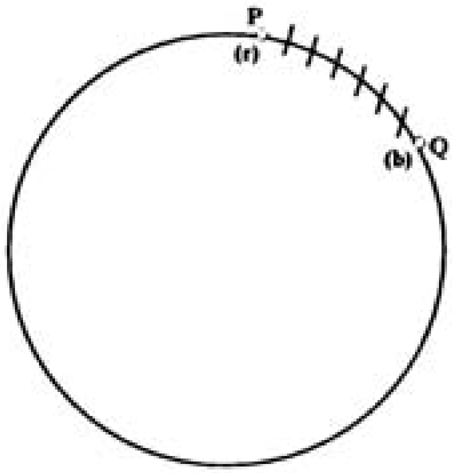


FIGURE 77

must join two blue points, showing that B does indeed determine a chord of every possible length d.

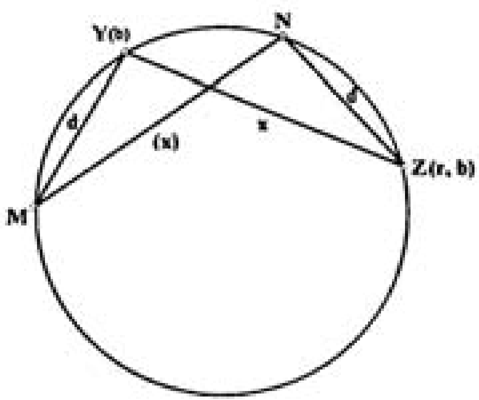


FIGURE 78

This problem and a wealth of results in combinatorial geometry, including the challenging exercise below (with solution), are given in the truly outstanding book *Combinatorial Geometry in the plane*, by Hadwiger, Debrunner, and Klee.

Exercise

If each point of a unit segment is colored red or blue so that each colored set is closed, then at least one of the colored sets must determine a segment of every length up to and including $\frac{1}{3}$, but not necessarily of *every* length up to any size greater than $\frac{1}{3}$.

From the 1987 Austrian-Polish Team Competition

(*Crux Mathematicorum*, 1988, 36)

Positive integers which read the same backwards as forwards are called *palindromes*; for example 55, 898, and 30103. In this problem we are concerned with the subset N of the palindromes in which each member is distinguished by the (admittedly contrived) property that

> the product of its digits is three less than three times the sum of its digits: i.e., $p = 3s - 3$.

Two examples are 616 and 292, for both of which

$$p = 36 \quad \text{and} \quad s = 13.$$

Prove that N has an infinity of members but that the subset Q of N, consisting of the members which don't contain any 1's in them (like 292), is only finite, and find all members of Q.

Solution

Clearly p is divisible by 3, and so some digit of a member of N must be divisible by 3.

The beauty of using 1's, of course, is that they bolster the sum of the digits in a number under construction without affecting the product, and therefore can be used to raise a currently deficient sum to the required

$$s = \frac{p+3}{3}.$$

For example, in looking for members of N, suppose we had built an integer to the stage 535. In this case, the product p would stand currently at 75 and the current sum at 13. However, with $p = 75$, the demand that $s = \frac{p+3}{3}$ would require $s = \frac{78}{3} = 26$, showing that the current sum is in need of another 13 units to bring it up to strength. The trouble is that 13 is an odd number, making it impossible to put $6\frac{1}{2}$ 1's at each end of 535 to preserve its palindromic character. However, this is only a minor setback, for if we start off with a central portion that contains an *even* number of equal digits, like 5335, it wouldn't matter how many 1's might be required to raise the sum to the required value: if it's an even number, put half of them at each end; if an odd number, put one right in the middle between the two 3's and half of the rest at each end. In the case of 5335, we have

$$p = 225, \quad \text{requiring} \quad s = \frac{228}{3} = 76;$$

with the current sum at 16, then, another 30 1's at each end will do the trick; hence

$$\underbrace{11\ldots\ldots\ldots1}_{30}\,5\,3\,3\,5\,\underbrace{1\ldots\ldots\ldots1}_{30} \in N.$$

Of course, this procedure only works when the current sum needs increasing. But this is easily arranged. For example, with 4-digit numbers, the current sum cannot exceed $4 \cdot 9 = 36$, and so the product p need only exceed $3 \cdot 36 - 3 = 105$ in order to get the process going (as it does in the case of 5335 and many other obvious choices). It is quite evident, then, that N is an infinite set. However, an actual proof along these lines would require a formula that generates an infinity of members of N. Again, this is no problem, for there are any number of formulas we might use.

Sticking with 5's and 3's, consider the number with k 5's on each side of two 3's:

$$\underbrace{55\ldots\ldots\ldots5}_{k}\,3\,3\,\underbrace{5\ldots\ldots\ldots5}_{k}.$$

In this case, $p = 9 \cdot 5^{2k}$, requiring $s = 3 \cdot 5^{2k} + 1$. Thus the current sum needs increasing by

$$3 \cdot 5^{2k} + 1 - (10k + 6) = 3 \cdot 5^{2k} - 10k - 5,$$

an even number. Accordingly, for all $k = 1, 2, 3, \ldots$,

$$\underbrace{11\ldots\ldots\ldots\ldots\ldots\ldots1}_{\frac{1}{2}(3\cdot 5^{2k} - 10k - 5)}\;\underbrace{55\ldots\ldots5}_{k}\,3\,3\,\underbrace{5\ldots\ldots5}_{k}\;\underbrace{1\ldots\ldots\ldots\ldots\ldots\ldots1}_{\frac{1}{2}(3\cdot 5^{2k} - 10k - 5)} \in N$$

For $k = 2$, we get

$$\underbrace{11\ldots\ldots\ldots1}_{925}\,5\,5\,3\,3\,5\,5\,\underbrace{1\ldots\ldots\ldots\ldots1}_{925} \in N$$

$$(p = 9\cdot 5^4 = 5625, \text{and } s = 1876).$$

On the other hand, members of Q, not having any 1's, can't have very many digits. If a member of Q has n digits, then, recalling some digit is divisible by 3, we have

$$p \geq 3\cdot 2^{n-1} \quad \text{and} \quad s \leq 9n.$$

Hence

$$27n - 3 \geq 3s - 3 = p \geq 3\cdot 2^{n-1},$$

and

$$9n - 1 \geq 2^{n-1},$$

which certainly fails for $n \geq 7$ (an easy exercise by induction). Therefore no member of Q can have more than 6 digits, implying that Q is finite alright.

As far as I can tell, the rest of the problem reduces to the checking of cases. Perhaps that's what makes it a good team question—while one member is checking 3-digit numbers another can be checking 4-digit numbers, and so on. In any case, according to my figuring, the only member of Q is 292. Since the details are somewhat tedious, let's move on to something more exciting.

Two Problems from the 1987 Austrian-Polish Mathematics Competition

(*Crux Mathematicorum*, 1987, 35. An alternative solution is given in *Crux Mathematicorum*, 1989, 269; comment 1990, 101.)

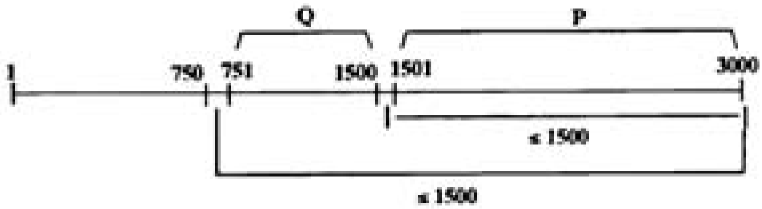
Problem 1

Does the set $X = \{1, 2, \ldots, 3000\}$ contain a subset A of 2000 integers in which no member of A is twice another member of A?

Solution

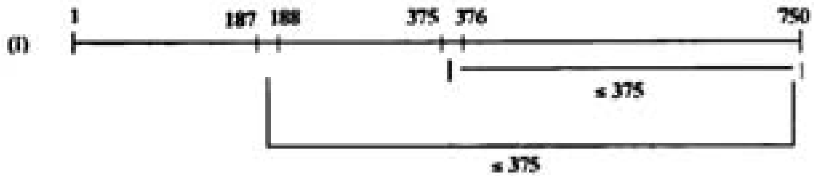
Since twice any integer in the interval $P = [1501, 3000]$ is too big to belong to X, we could put these 1500 integers in A without worrying about doubling up on any of them. On the other hand, A certainly can't get more than 1500 integers from P since it only has 1500 altogether. Obviously, we have to be careful not to pick any integer in the interval $Q = [751, 1500]$ which is one-half an integer that is chosen from P. Clearly, each integer taken from Q negates the eligibility of its double in P, and it follows that, if k integers are taken from Q, then not more than $1500 - k$ can be selected from P, for a total of not more than 1500 altogether from the entire interval $Q \cup P = [751, 3000]$. Thus, in order to build up to 2000 integers in A, at least 500 must come from $[1, 750]$, the initial quarter of X (Figure 79).

Repeated applications of this reasoning give the following results.

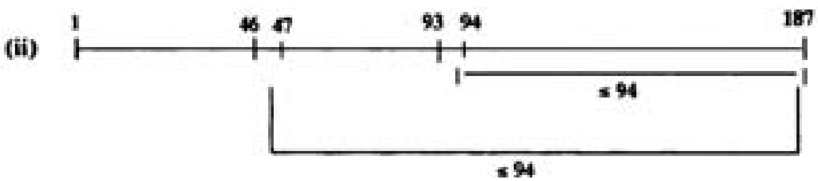


$\Longrightarrow$ $\geq$ 500 from [1,750]

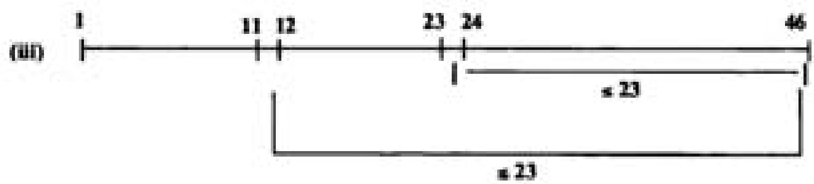
FIGURE 79



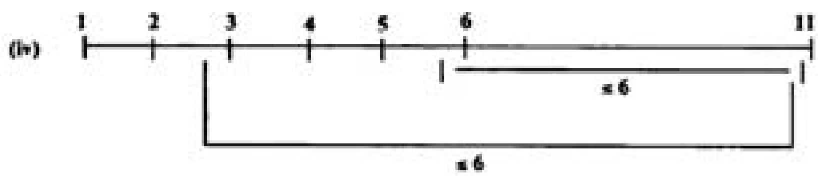
$\Longrightarrow$ $\geq$ 125 from [1,187]



$\Longrightarrow$ $\geq$ 31 from [1,46]



$\Longrightarrow$ $\geq$ 8 from [1,11]



$\Longrightarrow$ $\geq$ 2 from [1,2],

which shows the undertaking is hopeless.

FIGURE 80(i–iv)

However, the proposal is not outrageous, for clearly A can be built up to

$$1500 + 375 + 94 + 23 + 6 + 1 = 1999 \quad \text{integers.}$$

This suggests that two-thirds the number of integers in X is a sharp cut-off point for the size of A, that is, that $|A|$ can be any number up to two-thirds the size of X, but not actually as big as $\frac{2}{3}|X|$. Applying our analysis to $X = \{1, 2, \ldots, 300\}$, however, reveals that A *can* have as many as 200 members:

$$A = \{\underbrace{1, 3, 4;}_{3} \quad \underbrace{10, 11, \ldots, 18;}_{+\ \ 9} \quad \underbrace{38, 39, \ldots, 75;}_{+\ \ 38} \quad \underbrace{151, 152, \ldots, 300}_{+\ \ 150 = 200}\}$$

But, this set is as fully packed as possible, suggesting that the general result is rather $|A| \leq \frac{2}{3}|X|$.

It is a pleasure to report that Bruce Resnick (University of Illinois at Urbana-Champaign) has recently found the following pretty formula for the maximum size $f_r(n)$ of a subset of $\{1, 2, \ldots, n\}$ in which no element is r times another. Converting n to its representation in base r,

$$n = a_m a_{m-1} \cdots a_0,$$

then

$$f_r(n) = \frac{1}{r+1}\left(rn + \sum_{k=0}^{m} (-1)^k a_k\right).$$

Since $n = 3000$ is 101110111000 in base 2, it follows that

$$\begin{aligned} f_2(3000) &= \tfrac{1}{3}[2 \cdot 3000 + (-1 + 1 - 1 - 1 + 1 - 1 - 1)] \\ &= \tfrac{1}{3}(6000 - 3) \\ &= 1999, \quad \text{as found above.} \end{aligned}$$

Problem 2

Suppose the points of three-dimensional space are partitioned into three nonempty subsets A_1, A_2, A_3. Prove that the points of at least one of the subsets must determine all possible distances, that is, prove that one of the subsets A_i is such that, for each positive real number d, some two points of A_i are at a distance d from each other.

Solution

Clearly there are only two possibilities: either

(i) each of A_1, A_2, A_3 fails to realize all distances, or
(ii) at least one of them doesn't fail.

Since we would like to establish (ii), let's try to show that (i) leads to a contradiction.

Suppose, then, that

(a) A_1 fails to realize the distance d_1,
(b) A_2 fails to realize the distance d_2, and
(c) A_3 fails to realize the distance d_3.

Without loss of generality, we may suppose that $d_1 \geq d_2 \geq d_3$.

Let X be any point of A_1 and let S be the sphere with center X and radius d_1 (figure 81). If any point on S were to belong to A_1, then A_1 would generate the distance d_1, contrary to assumption (a). Thus each point of S must belong to A_2 or A_3.

If every point of S were to belong to A_3, then, because $d_3 \leq d_1$, A_3 would realize the distance d_3, contradicting assumption (c) (the chords of a great circle on S determine all distances up to $2d_1$, which is greater than d_3). Hence some point Y of S must belong to A_2.

Now, since $d_2 \leq d_1$ = the radius of S, a sphere of radius d_2, centered at Y, would intersect S in a circle K, no point of which could belong to A_2 without contradicting assumption (b). Because each point of S belongs either to A_2 or A_3, then K must belong entirely to A_3.

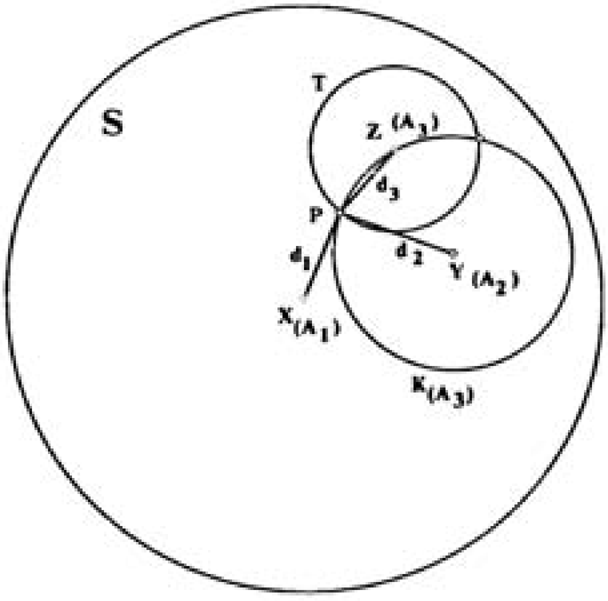


FIGURE 81

Finally, a sphere T of radius d_3, centered at any point Z of K intersects K in a point P which,

being on S, is a distance d_1 from X,
being on K, is a distance d_2 from Y, and
being on T, is a distance d_3 from Z.

Thus, no matter which subset P might be in, it violates one of our assumptions, and the argument is complete.

An Engaging Property Concerning the Incircle of a Triangle

(Problem P1245, *Crux Mathematicorum*, 1988, 189)

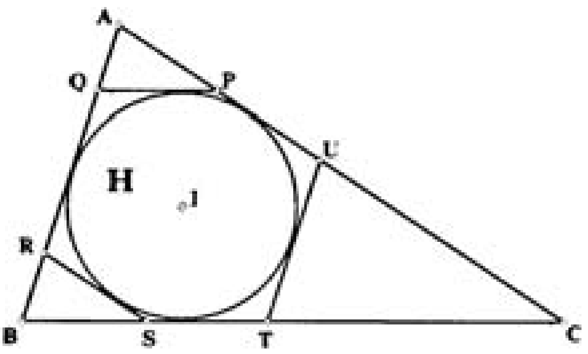


FIGURE 82

Let $H = PQRSTU$ be the hexagon determined in $\triangle ABC$ by drawing tangents to the incircle that are parallel to the sides of the triangle. Prove that the perimeter of H is never more than two-thirds that of $\triangle ABC$.

This problem comes from Walther Janous, Ursulinengymnasium, Innsbruck, Austria, and the solution is due to Hans Engelhaupt, Gundelsheim, The Federal Republic of Germany.

Solution

Since a casual glance at Figure 82 gives the impression that the sides of H are of irregular lengths and meet at undistinguished angles, it is easy to overlook the

fact that the opposite sides of H are equal. It is simple enough to convince oneself that this is true by considering a half-turn about the incenter I. For example, the sides RQ and QP are carried to lie along the parallel *lines* which contain the opposite sides UT and TS, and hence their point of intersection Q must be carried to the intersection T of these lines. Similarly P is carried to S, and we have $QP = TS$.

If the lengths of the tangents are $PQ = a'$, $RS = b'$, and $TU = c'$, then $a' + b' + c'$ is the semiperimeter of H and the desired ratio L is

$$L = \frac{\text{perimeter of } H}{\text{perimeter of } \triangle ABC} = \frac{\text{semiperimeter of } H}{\text{semiperimeter of } \triangle ABC} = \frac{a' + b' + c'}{s}.$$

If $AD = h_a$ is the altitude from A (see Figure 83), then, because QP is parallel to BC, triangle AQP is similar to $\triangle ABC$, and we have

$$\frac{QP}{BC} = \frac{a'}{a} = \frac{\text{altitude } AE}{\text{altitude } AD} = \frac{h_a - DE}{h_a} = \frac{h_a - 2r}{h_a} = 1 - \frac{2r}{h_a}.$$

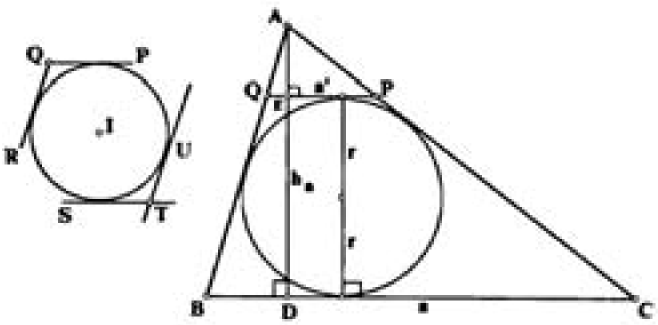


FIGURE 83

From the well-known formula for the area of a triangle $\triangle = rs$, we have

$$\frac{1}{2} a h_a = rs, \quad \text{giving} \quad \frac{2r}{h_a} = \frac{a}{s}.$$

Hence

$$\frac{a'}{a} = 1 - \frac{a}{s},$$

and

$$a' = a - \frac{a^2}{s}.$$

With similar expressions for b' and c', then, the required ratio is

$$L = \frac{a - \dfrac{a^2}{s} + b - \dfrac{b^2}{s} + c - \dfrac{c^2}{s}}{s} = \frac{a + b + c - \dfrac{1}{s}(a^2 + b^2 + c^2)}{s},$$

and since $a + b + c = 2s$, we have

$$L = 2 - \frac{a^2 + b^2 + c^2}{s^2} = 2 - \frac{4(a^2 + b^2 + c^2)}{(a + b + c)^2}.$$

It remains to get a bound on $\dfrac{(a^2 + b^2 + c^2)}{(a + b + c)^2}$.

At this point, Hans Engelhaupt craftily observes that a sum of squares is nonnegative, and therefore

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0.$$

Expanding and transposing, we get

$$\begin{aligned} 2(a^2 + b^2 + c^2) &\geq 2ab + 2bc + 2ac \\ &= (a + b + c)^2 - (a^2 + b^2 + c^2), \end{aligned}$$

giving

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2,$$

and

$$\frac{a^2 + b^2 + c^2}{(a + b + c)^2} \geq \frac{1}{3}.$$

(This can also be obtained immediately from the power mean inequality.) Hence

$$L \leq 2 - 4\left(\frac{1}{3}\right) = \frac{2}{3},$$

as desired.

On Floors and Ceilings

(Problem 1081, *Crux Mathematicorum*, 1987, 93)

Non-integral real numbers often crop up in work in which we are interested only in integers. For example, if n is known to be an integer, the restriction $n \leq 6.25$ is no better than $n \leq 6$; similarly, $n \geq 6.25$ tells us no more than $n \geq 7$. In combinatorics, one is forever rounding real numbers x either up or down. The results of doing this are called the *floor* and *ceiling* functions of x and are denoted by

$$[x] \quad \text{or} \quad \lfloor x \rfloor = \text{floor of } x = \text{the greatest integer } \leq x \text{ (round down)},$$
$$\lceil x \rceil = \text{ceiling of } x = \text{the least integer } \geq x \text{ (round up)}.$$

Thus

$$\lfloor 6.25 \rfloor = 6, \quad \text{and} \quad \lceil 6.25 \rceil = 7.$$

There are many interesting problems involving $\lfloor x \rfloor$ and $\lceil x \rceil$, some rather disconcerting because we are so fond of dealing with the continuity of the real numbers. In any case, I hope you will enjoy the following delightful problem of Loren Larson (St. Olaf College, Northfield, Minnesota). The solution is due to Svi Margaliot, at the time a grade 11 student at A. B. Lucas Secondary School, London, Ontario.

For a given integer $b > 1$, what is the value of the integral

$$I = \int_0^\infty \left\lfloor \log_b \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor \right\rfloor \mathrm{d}x?$$

Solution

Perhaps we can take some comfort in the fact that the integrand $e(x)$ is always an integer, for this signals a step-function of some kind. Since it is always a good idea to familiarize ourselves with a new function by working out a few of its values, let's see what its value is at $x = 6.25$, for example. In this case,

$$\lceil x \rceil = 7, \quad \frac{\lceil x \rceil}{x} = \frac{7}{6.25},$$

a number between 1 and 2, giving $\left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor = 1$, and then, no matter what b is,

$$\log_b 1 = 0,$$

implying

$$\lfloor \log_b 1 \rfloor = 0 \quad \text{and} \quad e(6.25) = 0.$$

As a matter of fact, any $x \geq 1$ gives the same result, for the rounded-up value of every $x \geq 1$ is never as much as $2x$, and so $\lceil x \rceil / x$ lies between 1 and 2 (possibly equal to 1 but always less than 2), making its floor equal to 1, the logarithm equal to 0, and $e(x) = 0$. Thus we can disregard all $x \geq 1$, for

$$I = \int_0^\infty e(x)\,\mathrm{d}x = \int_0^1 e(x)\,\mathrm{d}x.$$

We should be alert to the fact that the integrand is not defined at $x = 0$ ($\lceil 0 \rceil / 0 = 0/0$), implying the integral is improper and actually given by

$$I = \lim_{a \to 0^+} \int_a^1 e(x)\,\mathrm{d}x.$$

However, we needn't be concerned about having to deal with a nagging limit process for, as we shall see, this discontinuity at $x = 0$ presents no problem at all.

For all positive $x < 1$, clearly $\lceil x \rceil = 1$, and hence

$$\frac{\lceil x \rceil}{x} > 1, \quad \text{making} \quad \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor \quad \text{a positive integer.}$$

Since the base-b logarithms of all integers between b^k and b^{k+1} lie between k and $k+1$, they all round down to the same integer k. That is to say, for all x such that

$$b^k \leq \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor < b^{k+1}, \text{ i.e.,} \quad \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor = b^{k.abc\ldots},$$

the integrand

$$e(x) = \left\lfloor \log_b \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor \right\rfloor = \lfloor \log_b (b^{k.abc\ldots}) \rfloor = \lfloor k.abc\ldots \rfloor = k.$$

Thus $e(x) = k$ for all x that satisfy the inequalities

$$b^k \leq \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor < b^{k+1}.$$

Now, y cannot exceed $\lfloor y \rfloor$ by as much as 1, and so if $a \leq \lfloor y \rfloor < b$, where a and b are integers, then also $a \leq y < b$. Hence these inequalities are equivalent to

$$b^k \leq \frac{\lceil x \rceil}{x} < b^{k+1}.$$

Recalling that $\lceil x \rceil = 1$ for all x in the interval of integration (0,1], these conditions are met by all x which satisfy

$$b^k \leq \frac{1}{x} < b^{k+1},$$

that is to say, for all x in the interval

$$\frac{1}{b^k} \geq x > \frac{1}{b^{k+1}}.$$

Thus, with proper regard for all the endpoints of the interval, as x runs along the axis toward the origin from $\frac{1}{b^k}$ to $\frac{1}{b^{k+1}}$, every value of $e(x) = k$:

as x goes from 1 down to $\frac{1}{b}$, $\qquad e(x) = 0$,

as x goes from $\frac{1}{b}$ down to $\frac{1}{b^2}$, $\qquad e(x) = 1$,

as x goes from $\frac{1}{b^2}$ down to $\frac{1}{b^3}$, $\qquad e(x) = 2$,

. . . .

Hence $e(x)$ is the step-function pictured in Figure 84.

The integral I, being the area under the graph of $e(x)$, is most easily calculated by partitioning the region into unit-high rectangles $R_1, R_2, \ldots$, with bases $\frac{1}{b}, \frac{1}{b^2}, \frac{1}{b^3}, \ldots$, as shown.

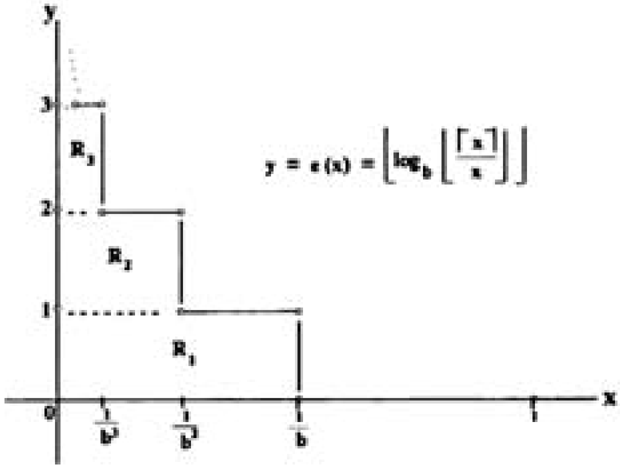


FIGURE 84

Hence

$$I = \int_0^1 e(x)\,\mathrm{d}x = \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \cdots,$$

where the limit process arising from the discontinuity at $x = 0$ is accommodated simply by letting the series go on indefinitely. Finally, then,

$$I = \frac{\dfrac{1}{b}}{1 - \dfrac{1}{b}} = \frac{1}{b-1}.$$

Two Problems from the International Olympiad 1987

(*Crux Mathematicorum*, 1987, 210)

Problem 1

For every integer $n \geq 3$, prove there exists in the plane a set of n points such that

(a) the distance between each two of the points is *irrational*, and
(b) each three of the points determine a *nondegenerate* triangle having *rational* area.

I don't have any idea how the examiners felt the contestants might approach this problem, and I can only account for my own solution as largely a matter of good luck.

Solution

Since each trio of the points is to determine a nondegenerate triangle, it is clear that no three of them can lie in a straight line. Thus the possibility arises that such a set of points might be found strung out along a well-known curve, like a circle. However, the usual representation of points on a circle, $(r\cos\theta, r\sin\theta)$, seemed likely to lead to complicated expressions for the distances and areas, and while casting about for a curve with simpler parameters, my favorite parabola $y^2 = x$ came to mind. It was a simple matter to check out the points $A(k^2, k)$, $B(t^2, t)$, and $C(q^2, q)$ on this curve and, as luck would have it, everything worked out surprisingly quickly and easily.

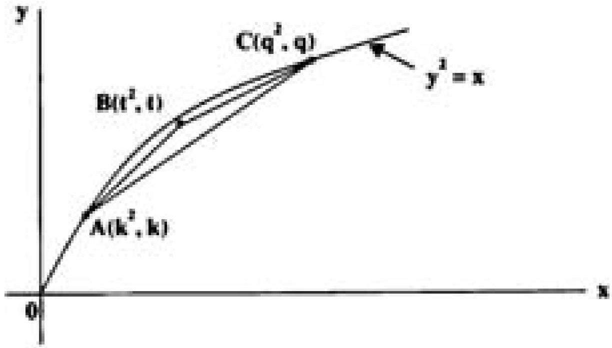


FIGURE 85

Clearly

$$AB = \sqrt{(k^2 - t^2)^2 + (k - t)^2}$$
$$= (k - t)\sqrt{(k + t)^2 + 1},$$

which is always irrational for unequal positive integers k and t ($(k + t)^2$ *is* a perfect square, and so the consecutive integer $(k + 1)^2 + 1$ cannot be a perfect square). And since the determinant

$$D = \begin{vmatrix} k^2 & k & 1 \\ t^2 & t & 1 \\ q^2 & q & 1 \end{vmatrix}$$

is an *integer* for integers k, t, and q, the area of $\triangle ABC$, which is just $|\frac{1}{2}D|$, is always rational.

Therefore an acceptable set of n points is simply

$$\{(1^2, 1), (2^2, 2), (3^2, 3), \ldots, (n^2, n)\}.$$

Problem 2

Let the bisector of angle A in acute triangle ABC cross BC at L and meet the circumcircle of the triangle at N. From L, let perpendiculars LK and LM be drawn to the sides AB and AC, respectively. Prove that quadrilateral $AKNM$ and triangle ABC have the same area:

$$AKNM = \triangle ABC.$$

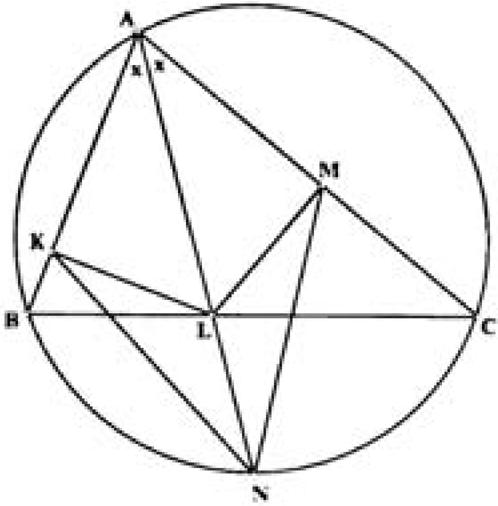


FIGURE 86

Solution

Perhaps the first thing to strike us is that triangles AKL and AML are congruent (angle-angle-side), giving

$$KL = LM.$$

Since KL and LM are the altitudes of the triangles into which $\triangle ABC$ is divided by AL, we have

$$\begin{aligned}\triangle ABC &= \frac{1}{2}AB \cdot KL + \frac{1}{2}AC \cdot LM \\ &= \frac{1}{2}c \cdot KL + \frac{1}{2}b \cdot KL \\ &= \frac{1}{2}(c+b)KL.\end{aligned}$$

Now, the congruent triangles also give $AK = AM$, and so AL is the bisector of the vertical angle in isosceles triangle AKM. Hence AL is actually the perpendicular bisector of the base KM, and accordingly it crosses it at right angles at its midpoint Q (Figure 87). Thus $KQ = QM$, and

$$\begin{aligned}AKNM &= \triangle AKN + \triangle AMN \\ &= \frac{1}{2}AN \cdot KQ + \frac{1}{2}AN \cdot QM \\ &= AN \cdot KQ.\end{aligned}$$

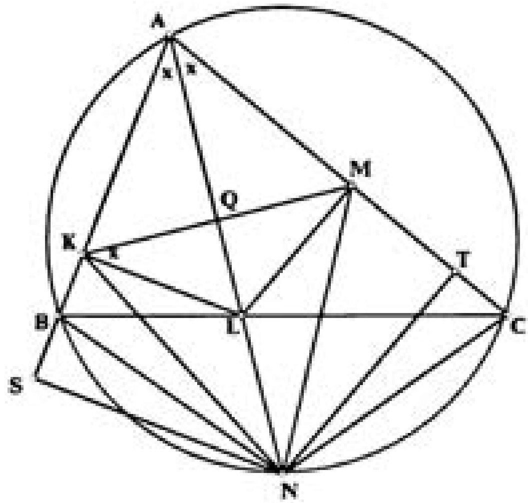


FIGURE 87

Now, the right angles at K and M make $AKLM$ cyclic, and on the chord LM in its circumcircle,

$$\measuredangle LKM = \measuredangle LAM = \frac{1}{2}\measuredangle A = x.$$

Thus, in right triangle KLQ,

$$KQ = KL\cos x,$$

and we have

$$AKNM = AN \cdot KL\cos x.$$

We wish to show, then, that

$$\triangle ABC = \frac{1}{2}(c+b)KL = AN \cdot KL\cos x,$$

that is,

$$b + c = 2AN\cos x.$$

Now, $AN\cos x$ is obviously given by the side AT of the right triangle obtained by dropping a perpendicular NT to AC, and it is equally well given by the side AS of right triangles NAS (Figure 87). Hence

$$2AN\cos x = TA + AS,$$

and we would like to show that

$$TA + AS = b + c.$$

Now,

$$b + c = CA + AB,$$

and if this is to be the same as $TA + AS$, then CT would have to equal BS; and conversely, if we can show that $CT = BS$, the desired conclusion would follow. But it is easy to see that $CT = BS$, for they are corresponding sides in congruent right triangles NCT and NBS:

> the hypotenuses NC and NB are equal chords in the circle because they subtend equal angles at A, and $NT = NS$ in congruent triangles NAS and NAT (AAS).

On Arithmetic Progressions

(P1166 (restated), *Crux Mathematicorum*, 1987, 324, proposed by Kenneth S. Williams, Carleton University, Ottawa, Canada)

Some arithmetic progressions of positive integers contain perfect squares and others don't. For example,

$$4, 10, 16, 22, \ldots, \text{ clearly has 4 and 16, but}$$

$$3, 10, 17, 24, \ldots, \text{ has no perfect square at all.}$$

Prove that if $a, a+d, a+2d, \ldots$ is going to contain a square, then one must show up before the progression reaches the number

$$N = a + 2d\sqrt{a} + d^2.$$

Solution

The following incisive solution is due to Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We are required to show that if the progression contains any square k^2 whatever, it must contain one which is $< N$. Observing that

$$N = a + 2d\sqrt{a} + d^2 = (\sqrt{a} + d)^2,$$

this requires a term $x^2 < (\sqrt{a} + d)^2$, and $x < \sqrt{a} + d$.
Suppose, then, that some term does equal k^2: i.e.,

$$a + nd = k^2 \quad \text{for some} \quad n \geq 0.$$

Clearly this gives

$$k^2 \equiv a \pmod{d}.$$

Now, the interval $[\sqrt{a}, \sqrt{a} + d)$ has length d, and being closed at one end, it must contain d positive integers $\{t+1, t+2, \ldots, t+d\}$. (If $\sqrt{a}$ is an integer, then the string starts at $t + 1 = \sqrt{a}$.) Since any d consecutive integers constitute a complete set of residues modulo d, the integer k must be congruent to some integer x in this interval:

$$x \equiv k \pmod{d}.$$

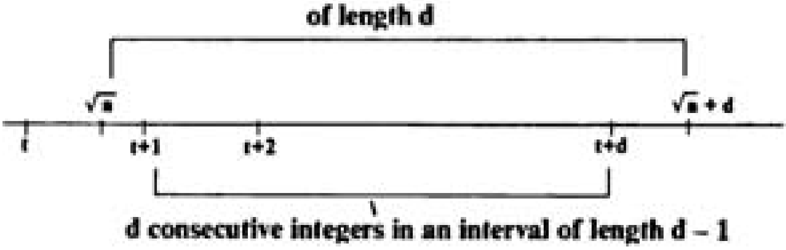


FIGURE 88

Thus

$$x^2 \equiv k^2 \equiv a \pmod{d},$$

and for some $n \geq 0$, it follows that

$$x^2 = a + nd.$$

That is to say, x^2 *is* a term in the progression, and being in the interval $[\sqrt{a}, \sqrt{a} + d)$,

$$x < \sqrt{a} + d,$$

giving the desired

$$x^2 < (\sqrt{a} + d)^2 = N.$$

For the given example,

$$3, 10, 17, 24, 31, 38, 45, 52, 59, 66, 73, 80, \ldots,$$
$$N = 3 + 14\sqrt{3} + 49 < 77,$$

and since it contains no square $< N$, it can contain no square at all.

A Property of Triangles Having an Angle of 30°

(P1100, *Crux Mathematicorum*, 1987, 160)

As usual, let I and O be the incenter and circumcenter, respectively, of $\triangle ABC$. Suppose $\measuredangle C = 30°$, and that the side AB is laid off along each of the other two sides to give points D and E so that

$$EA = AB = BD.$$

Prove that the segment DE is both equal and perpendicular to IO.

This problem is due to D.J. Smeenk, Zaltbommel, The Netherlands, and the following brilliant solution is by Hidetosi Fukagawa (Yokosuka High School, Tokai-city, Aichi, Japan), whose book *Japanese Temple Geometry Problems*, written in collaboration with Dan Pedoe (Minneapolis, Minnesota), is a historic landmark.

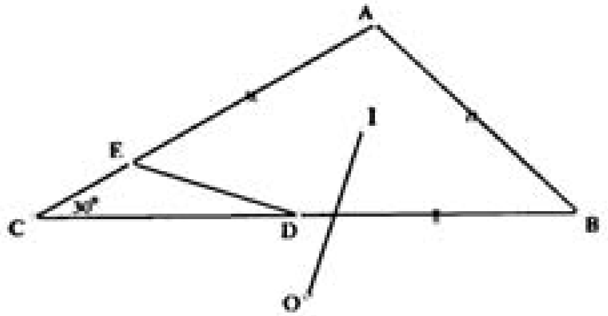


FIGURE 89

Solution

Let AI be extended to meet the circumcircle of $\triangle ABC$ at F. Because I is the incenter, IA is the bisector of $\measuredangle A$, and since $\triangle EAB$ is isosceles, this bisector meets the base at its midpoint M, and is in fact the perpendicular bisector of the base EB. The point F on this perpendicular bisector is therefore equidistant from E and B:

$$EF = FB.$$

Also, $\triangle EFA \equiv \triangle BFA$ (SAS), making $\measuredangle EFA = \measuredangle BFA$, and implying $\measuredangle EFB = 2\measuredangle AFB$. But $\measuredangle AFB = \measuredangle C$ on chord AB, and so

$$\measuredangle EFB = 2(30^\circ) = 60^\circ.$$

Triangle EFB, then, has a 60° angle between its equal sides EF and FB, and therefore is an **equilateral** triangle. Hence $FB = EB$.

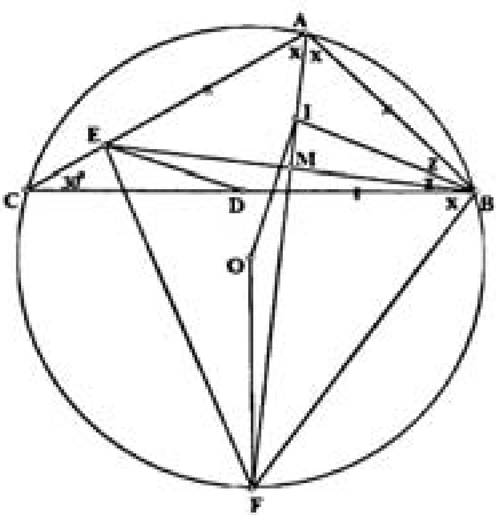


FIGURE 90

Now, it is not difficult to see that $\triangle IBF$ is isosceles, for the angles at I and B are each $\frac{1}{2}A + \frac{1}{2}B$:

$$\measuredangle CAF = \measuredangle CBF \text{ on chord } CF, \text{ making } \measuredangle CBF = \frac{1}{2}A;$$

and since IB bisects $\measuredangle B$, we have $\measuredangle IBF = \frac{1}{2}B + \frac{1}{2}A$; but, from $\triangle AIB$, exterior angle FIB also equals $\frac{1}{2}A + \frac{1}{2}B$. Therefore $FB = FI$, and since $FB = EB$ in equilateral $\triangle EBF$, we have

$$FI = BE. \tag{1}$$

Now let's use this to prove triangles *FIO* and *BED* are congruent, from which our desired conclusion follows quickly. Since *AI* bisects $\measuredangle A$, *F* must be the midpoint of arc *CB*, implying that *FO* is perpendicular to the chord *CB*. Recalling that *IF* is the perpendicular bisector of *EB*, we have the arms of angles *IFO* and *EBD* respectively perpendicular, and it follows, then, that they are equal:

$$\measuredangle OFI = \measuredangle EBD. \tag{2}$$

Now, *AB* subtends at the center *O* twice the angle it subtends at *C* on the circumference, implying that $\measuredangle AOB = 2\measuredangle C = 60^\circ$. The equal radii *OA* and *OB*, then, also make $\triangle AOB$ equilateral, and we conclude that the circumradius is none other than *AB* itself. Consequently,

$$\text{radius } OF = AB = BD. \tag{3}$$

Hence triangles *OFI* and *EBD* have two sides and the contained angles respectively equal (from (1), (2), and (3), above), making them congruent, and giving $IO = ED$.

But this also gives $\measuredangle FIO = \measuredangle BED$. And since we have already seen that the arms *IF* and *EB* of these equal angles are perpendicular, their other arms *IO* and *ED* must also be perpendicular: rotating *IF* and *EB* about *I* and *E*, respectively, through the same angle in the same direction (clockwise in the figure) to the other arms *IO* and *ED* preserves the perpendicularity.

From the 1985 Bulgarian Spring Competition (Grade 11)

(*Crux Mathematicorum*, 1987, 138)

If

$$S_n = \sum_{k=0}^{n} \binom{3n}{3k} = \binom{3n}{0} + \binom{3n}{3} + \binom{3n}{6} + \cdots + \binom{3n}{3n},$$

prove that

$$\lim_{n\to\infty} S_n^{1/3n} = 2.$$

Solution

Clearly S_n is the sum of every third binomial coefficient in the series

$$(1+x)^{3n} = \binom{3n}{0} + \binom{3n}{1}x + \binom{3n}{2}x^2 + \cdots + \binom{3n}{3n}x^{3n}.$$

For $x = 1$, we get the sum of *all* these coefficients, and the problem is how to get rid of the two-thirds of them we don't want.

It is clear enough how to "bisect" a series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

From

$$f(1) = a_0 + a_1 + a_2 + a_3 + \cdots,$$

and

$$f(-1) = a_0 - a_1 + a_2 - a_3 + - \cdots,$$

we get

$$a_0 + a_2 + a_4 + \cdots = \frac{1}{2}\,[f(1) + f(-1)],$$

and

$$a_1 + a_3 + a_5 + \cdots = \frac{1}{2}\,[f(1) - f(-1)].$$

But it is not obvious how to trisect, or in general "multisect," a series.

However, we might notice that the values 1 and -1, which were substituted for x in the bisecting process, are none other than the two square roots of unity. Is there any chance the three cube roots of unity, 1, ω, and ω^2, where $\omega = e^{2\pi i/3}$, might serve us in trisecting a series? It seems unlikely, since two of these cube roots aren't even real. In any case, being the roots of the equation $x^3 - 1 = 0$, we know that

$$\omega^3 = 1, \quad \omega^{3k} = 1, \quad \text{and} \quad 1 + \omega + \omega^2 = 0.$$

Setting x equal to 1, ω, and ω^2, in turn, yields

$$(1+1)^{3n} = \binom{3n}{0} + \cdots + \binom{3n}{3k}\cdot 1 + \binom{3n}{3k+1}\cdot 1 + \binom{3n}{3k+2}\cdot 1 + \cdots,$$

$$(1+\omega)^{3n} = \binom{3n}{0} + \cdots + \binom{3n}{3k}\omega^{3k} + \binom{3n}{3k+1}\omega^{3k+1} + \binom{3n}{3k+2}\omega^{3k+2} + \cdots,$$

$$(1+\omega^2)^{3n} = \binom{3n}{0} + \cdots + \binom{3n}{3k}\omega^{6k} + \binom{3n}{3k+1}\omega^{6k+2} + \binom{3n}{3k+2}\omega^{6k+4} + \cdots.$$

Observing that

$$\omega^{3k} = \omega^{6k} = 1, \quad \omega^{3k+1} = \omega^{6k+1} = \omega,$$
$$\omega^{3k+2} = \omega^{6k+2} = \omega^2, \quad \text{and} \quad \omega^{6k+4} = \omega,$$

the *coefficients* of $\binom{3n}{3k}$, $\binom{3n}{3k+1}$ and $\binom{3n}{3k+2}$, in the *sum* of these series are, respectively,

$$1 + \omega^{3k} + \omega^{6k} = 3, \quad 1 + \omega^{3k+1} + \omega^{6k+2} = 1 + \omega + \omega^2 = 0,$$

and

$$1 + \omega^{3k+2} + \omega^{6k+4} = 1 + \omega^2 + \omega = 0.$$

Hence the sum $2^{3n} + (1+\omega)^{3n} + (1+\omega^2)^{3n}$ is simply $3S_n$, giving

$$S_n = \frac{1}{3}\,[2^{3n} + (1+\omega)^{3n} + (1+\omega^2)^{3n}].$$

Since $1+\omega+\omega^2=0$, we have $1+\omega=-\omega^2$ and $1+\omega^2=-\omega$, implying

$$\begin{aligned} S_n &= \frac{1}{3}[2^{3n}+(-1)^{3n}\omega^{6n}+(-1)^{3n}\omega^{3n}] \\ &= \frac{1}{3}[2^{3n}+(-1)^{3n}\cdot 2]. \\ &= \frac{1}{3}[2^{3n}\pm 2], \end{aligned}$$

as n is even or odd.

In all cases, then, we have

$$\frac{1}{3}(2^{3n}-2)\le S_n\le\frac{1}{3}(2^{3n}+2),$$

from which

$$\frac{1}{3}\left(1-\frac{2}{2^{3n}}\right)\le\frac{S_n}{2^{3n}}\le\frac{1}{3}\left(1+\frac{2}{2^{3n}}\right),$$

and

$$\left(\frac{1}{3}\right)^{1/3n}\left(1-\frac{2}{2^{3n}}\right)^{1/3n}\le\frac{S_n^{1/3n}}{2}\le\left(\frac{1}{3}\right)^{1/3n}\left(1+\frac{2}{2^{3n}}\right)^{1/3n}.$$

Now, for $0<|x|<1$, we have $\lim_{n\to\infty}x^{1/3n}=1$, and for $|x|>1$, we also have $\lim_{n\to\infty}x^{1/3n}=1$. Therefore, as $n\to\infty$, we get $\lim S_n^{1/3n}/2=1$, giving the desired $\lim S_n^{1/3n}=2$.

The above argument for trisecting a series extends directly to the general case of multisection. For a detailed proof of the general result, given below, see *Mathematical Gems III*, Vol. 9, Dolciani Series, MAA, 1985, page 210.

The General Formula

Let

$$f(x)=f_0+f_1x+f_2x^2+\cdots$$

be a finite or infinite series. Then the sum of every nth term, beginning with f_jx^j, $j\le n-1$, is given by

$$S(n,j)=\frac{1}{n}\sum_{t=0}^{n-1}\omega^{-jt}f(\omega^t x),$$

where $\omega=e^{2\pi i/n}$, an nth root of unity.

An Unused International Olympiad Problem from Britain

(*Crux Mathematicorum*, 1987, 77; solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.)

> Prove that the product of five consecutive positive integers is never a perfect square.

Solution

Since odd and even integers alternate in the sequence of integers, in any string of five consecutive positive integers, there must be at least two even integers, and possibly three. Similarly, any such string must contain at least one multiple of three, and possibly two. Finally, there must be exactly one multiple of five, and certainly not more than one multiple of any prime greater than 5.

Now, every positive integer n has a unique prime decomposition

$$n = 2^r 3^s 5^t 7^q \cdots .$$

Since only one of the five integers in the string is divisible by 5, a perfect square would be out of the question for their product unless the prime 5 were to occur to an *even* exponent in the decomposition of the integer which contains it. Similarly for any prime greater than 5. Since more than one integer in the string contributes 2's and 3's to the product, in any individual member of the string, the exponents r and s of 2 and 3 can be either odd or even. Consequently, if the product of the five integers in the string is to be a perfect square, *each* of the integers must have a prime decomposition

$$n = 2^r 3^s 5^{2a} 7^{2b} \ldots ,$$

where each of r and s can be odd or even, but every other exponent must be even. This permits into the string only integers of the following four kinds:

(i) r *even*, s *even*:

$$n = 2^{2k} 3^{2m} 5^{2a} 7^{2b} \cdots = (2^k 3^m 5^a 7^b \cdots)^2, \text{ a perfect square};$$

(ii) r *odd*, s *even*:

$$n = 2^{2k+1} 3^{2m} 5^{2a} \cdots = 2\,(2^k 3^m 5^a \cdots)^2, \text{ twice a perfect square};$$

(iii) r *even*, s *odd*:

$$n = 2^{2k} 3^{2m+1} 5^{2a} \cdots = 3\,(2^k 3^m 5^a \cdots)^2, \text{ 3 times a perfect square};$$

(iv) r *odd*, s *odd*:

$$n = 2^{2k+1} 3^{2m+1} 5^{2a} \cdots = 2 \cdot 3\,(2^k 3^m 5^a \cdots)^2, \text{ 6 times a perfect square}.$$

With only *four* possible kinds, the pigeonhole principle implies that some two of the *five* integers in the string must be the same kind. Now, the perfect squares are

$$1, 4, 9, 16, 25, \ldots,$$

and so twice a perfect square would be one of the numbers $2, 8, 18, \ldots$. But the difference between two numbers of this type is at least 6, implying that no two of these could both squeeze into a consecutive string of length 5. Similarly, no two of any of the types (ii), (iii), or (iv) could be present in our string:

three times a perfect square: $3, 12, 27, 48, \ldots$;
six times a perfect square : $6, 24, 54, 96, \ldots$.

Thus, if any string contains two integers of the same kind, they must be of type (i), i.e., both perfect squares. But the only string of length five that contains two perfect squares is 1, 2, 3, 4, 5, and since their product 120 is *not* a perfect square, there is no string at all that can meet the requirements.

A Rumanian Olympiad Proposal

Suppose A, B, and C are three different points on the same side of a plane π. The points A', B', and C' are chosen arbitrarily in π and the midpoints of AA', BB', and CC' are L, M, and N. S is the centroid of triangle LMN.

As A', B', C' vary in π, the point S moves accordingly. What is the locus of S as A', B', C' assume all possible 3-point configurations in π? (It is permissible for a pair or all three of A', B', C' to coincide.)

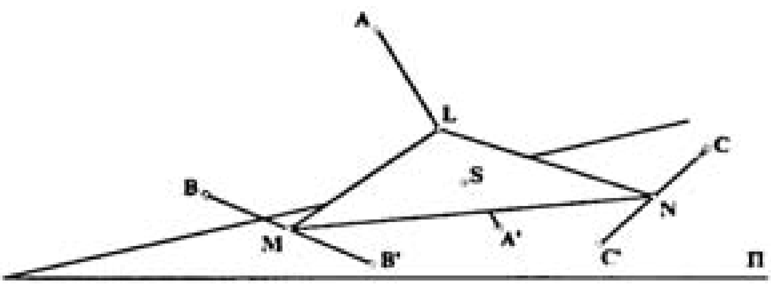


FIGURE 91

This problem, from Rumania, is given by Daniel Pedoe [1] as an illustration of the kind of geometry problem the eastern European nations felt was suitable for their secondary school mathematics competitions in the early 1960s. The following solution, which he suggests would likely be the approach that would be taken by a British student, is a very clever and beautiful argument.

Solution

If a particle of unit mass is suspended at each of A, B, C, A', B', C', the resulting system would be equivalent to a mass of 2 units at each of L, M, and N, and therefore also to a single mass of 6 units at S.

However, the center of mass S can also be determined in a second way. The three masses at A, B, and C are equivalent to a mass of 3 units at G, the centroid of $\triangle ABC$; similarly, the masses at A', B', and C' are equivalent to a mass of 3 at G', the centroid of $\triangle A'B'C'$. Since the masses at G and G' are equal, the center of mass S of the whole system is simply the midpoint of GG'.

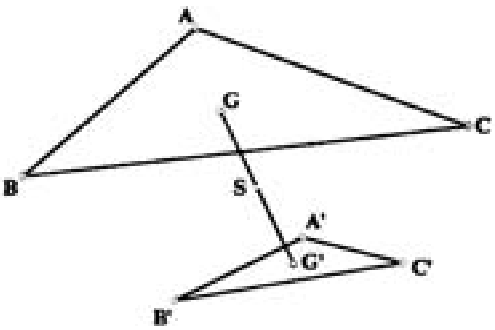


FIGURE 92

Now, as A', B', and C' roam around π, the points A, B, C, and more importantly G, remain fixed. Also, it is not difficult to convince oneself that as A', B', C' vary, G' takes on *all* positions in π (actually G' can be chosen first, and A', B', C' easily taken to suit). Thus the locus of S consists of the midpoints of all segments GG' as G' varies over π. Hence the locus is the image of the plane π under the dilatation $G(\frac{1}{2})$, which is simply the plane π' that is parallel to π and halfway between π and G.

Reference

[1] Daniel Pedoe, The Mathematical Tripos and Mathematical Education in Great Britain, *American Mathematical Monthly*, 1964, 666–670.

From the 1984 Bulgarian Olympiad

(*Crux Mathematicorum*, 1987, 288)

Solve

$$5^x \cdot 7^y + 4 = 3^z$$

for all nonnegative integral solutions (x, y, z).

Solution

The following brilliant solution is due to John Morvay of Dallas, Texas.

Since the left side is bigger than 3, the value of z must be at least 2. Thus each side of the equation is divisible by 3, and we have, modulo 3, that

$$5^x \cdot 7^y + 4 \equiv (-1)^x \cdot 1^y + 1 \equiv 0,$$

implying x must be odd. Hence x can't be zero, and considering the equation modulo 5, we have

$$5^x \cdot 7^y + 4 \equiv 4 \equiv 3^z.$$

Now, the powers of 3, reduced (mod 5), give the sequence of residues

$$\{3, 4, 2, 1, 3, 4, 2, 1, 3, \ldots\},$$

with period $(3, 4, 2, 1)$. In order to have $3^z \equiv 4 \pmod 5$, then, z must be $\equiv 2 \pmod 4$, that is, z must be one of the even numbers in the progression

$$\{2, 6, 10, 14, 18, \ldots\}.$$

For $z = 2$, the equation is

$$5^x \cdot 7^y + 4 = 9.$$

giving

$$5^x \cdot 7^y = 5,$$

and we obtain the solution $(x, y, z) = (1, 0, 2)$. However, as we shall see, this is the only solution.

For $z = 2k > 2$, we have

$$5^x \cdot 7^y + 4 = 3^{2k},$$

and

$$5^x \cdot 7^y = 3^{2k} - 4 = (3^k - 2)(3^k + 2), \quad \text{where} \quad k > 1.$$

Since the factors on the right differ by only 4, they can't both be divisible by 5, or both divisible by 7. Moreover, if both 5 and 7 were to divide the *same* factor, then the other factor would have to reduce to unity. Since this other factor would have to be the smaller one, this would lead to

$$3^k - 2 = 1, \qquad \text{and} \quad k = 1, \ \text{a contradiction.}$$

Hence 5^x must divide one of the factors and 7^y the other. In fact, since there can't be any other factors besides these, it must be that 5^x is equal to one of them and 7^y equal to the other. This requires that some power of 5 differ from some power of 7 by exactly 4. But we can easily see that this never happens, and so there is no other solution.

Being equal to one of the factors, we have, in any case, that

$$5^x \geq 3^k - 2 \geq 3^2 - 2 = 7 \quad (\text{recall } k > 1),$$

implying $x \geq 2$. Now, every power of 5, beyond the first, ends in the digits 25. A suitable power of 7, then would have to end in either 29 or 21. But the powers of 7 all end in either 07, 49, 43, or 01.

Two Erdös Problems

I learned long ago that confirming something that is obvious to Paul Erdös can entail a lengthy struggle in my mind. Even his simple-sounding problems are often deceptively deep and subtle. In any case, they are always interesting and frequently concern most striking properties. Here is a problem of his that appeared in 1950 in the *American Mathematical Monthly* (Problem 4330, page 493).

Problem 1

Prove that every infinite sequence S of distinct positive integers contains either

(a) an infinite subsequence such that, for every pair of terms, neither term ever divides the other, or
(b) an infinite subsequence such that, in every pair of terms, one always divides the other.

Solution

This solution is due to R. S. Lehman (Stanford University).

Let S be partitioned into two subsets A and B by putting into A all the terms which do *not divide* any other term of S. If A is an infinite set, then case (a) holds and the conclusion follows.

Suppose, then, that A is finite. In this case the complementary subset B must be finite (Figure 93). Now, there still might be some unwanted numbers cluttering up the set B. Let's remove into a subset C all the terms of B which

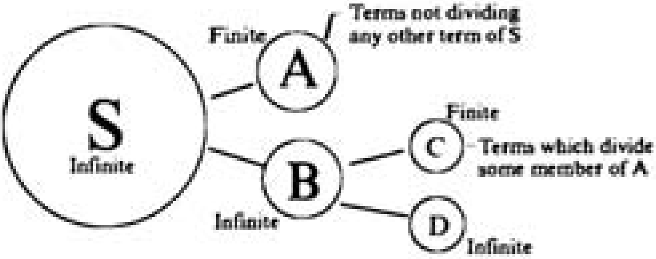


FIGURE 93

divide any member of A (being divisors, they fail to qualify for A itself, and therefore remain in B). Because A is finite, there is only a finite number of possible divisors of its members, making C a finite subset, possibly empty. Thus the complementary subset $D = B - C$, must be infinite.

Having got rid of the divisors of A, consider an integer k in D. Not being in A, k must divide at least one other term t of S. Such a term t cannot belong to A for, in that case, its divisor k would have been put into C, not left in D. Also, if t were to belong to C, then t would be a divisor of some member q of A, and then t's own divisor k would also be a divisor of q, again placing k in C, not D. Thus each member k of D must divide at least one other member t of D itself. Thus an infinite subsequence T can be constructed in an obvious way to have property (b), namely that, in every pair of terms, one always divides the other.

Beginning with any member k_1 of D, let $k_2 \in D$ be any multiple of k_1. Similarly, let $k_3 \in D$ be any multiple of k_2, and so on indefinitely. Because $k_1|k_2$ and $k_2|k_3$, we have $k_1|k_3$. With $k_3|k_4$, we obtain $k_1|k_4$, etc., leading to the conclusion that k_1 divides every term that follows it in T. But the same argument clearly establishes this property for every member k_i of T. Hence in any pair of terms (k_i, k_j), the earlier term always divides the later one.

Problem 2

(This is problem P1118 (revised) *Crux Mathematicorum*, 1987, 193; proposed and solved by Paul Erdős, Hungarian Academy of Sciences)

Let $a_1 < a_2 < a_3 < \cdots$ be an infinite sequence of positive integers in which the gaps between consecutive terms gets indefinitely large (perhaps not monotonically—gaps might get smaller before getting bigger, but eventually they do outrun all bounds). Given such a sequence $\{a_i\}$, determine how to construct a companion sequence of positive integers

$$b_1 < b_2 < b_3 < \cdots,$$

of infinite length, with the property that no finite subset of the b's ever adds up to any a_i.

Solution

We seek a sequence $\{b_i\}$ all of whose finite subsequences have sums that lie *between* the a_i, never equal to an a_i. For b_1, then, any positive integer that is not an a_i will do (since $a_{i+1} - a_i$ gets large, succeeding a_i can't remain *consecutive* integers forever, so there are lots of sizeable gaps from which to pick b_1). What we need is a general method of extending the sequence of b's that preserves the critical addition property.

Suppose $b_1 < b_2 < \cdots < b_k$ have already been successfully determined, i.e., all their subsets have sums in between the a_i. These k terms have some sum

$$S = b_1 + b_2 + \cdots + b_k,$$

and, no matter how big S might be, some gap $a_{n+1} - a_n$ is even bigger than $S + 1$: for some n,

$$a_{n+1} - a_n > S + 1,$$

giving

$$a_{n+1} > (a_n + 1) + S > a_n.$$

We shall see, then, that the string of b's can safely be extended by setting

$$b_{k+1} = a_n + 1.$$

Clearly this puts b_{k+1} itself between a_{n+1} and a_n:

$$a_n < a_n + 1 = b_{k+1} < b_{k+1} + S = (a_n + 1) + S < a_{n+1}.$$

And no matter what subset of $\{b_1, b_2, \ldots, b_k\}$ might be added to b_{k+1}, the sum will not reach a_{n+1} since

$$a_n < b_{k+1} + S < a_{n+1},$$

that is,

$$a_n < b_{k+1} + b_1 + b_2 + \cdots + b_k < a_{n+1}.$$

Thus, extending the b's to b_{k+1} in this way insures that the sum of all the *new* subsets of b's, which must involve b_{k+1} in order to be new, also have values *between* the a_i, as desired.

From the 1985 Bulgarian Olympiad

(*Crux Mathematicorum*, 1987, 39)

Let $P_1, P_2, \ldots, P_7$ be any seven points in space, no four in the same plane, and suppose that each of the $\binom{7}{2} = 21$ segments P_iP_j is colored either red or blue. Prove that, no matter how the colors may be distributed, two monochromatic triangles will be formed *which have no side in common*.

Solution

This is a golden opportunity to showoff a brilliant approach that employs "mixed angles," that is, angles with an arm of each color. Clearly the number of mixed angles in a monochromatic triangle is zero and the number in any other triangle is 2. Since no four of the given points are in the same plane, no two triangles can share an angle, and therefore the number of "mixed" triangles is simply one-half the total number of mixed angles. But it is not difficult to get an upper bound on the number of mixed angles.

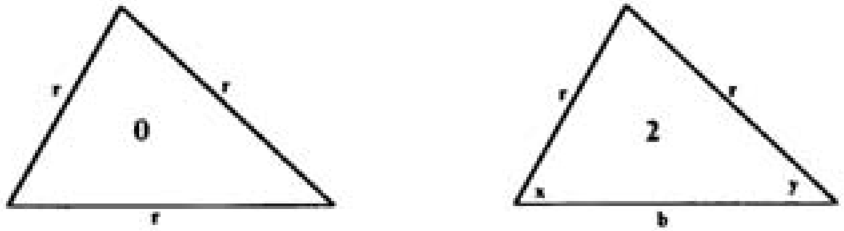


FIGURE 94

At each point P_i, there are six segments to the other P_j and the only possible ways the colors can split them are

(6,0) (i.e., 6 of one color and none of the other), (5,1), (4,2), and (3,3).

Since each red segment makes a mixed angle with each blue one, the number of mixed angles at each point is either

$$6 \cdot 0 = 0, \quad 5 \cdot 1 = 5, \quad 4 \cdot 2 = 8, \quad \text{or} \quad 3 \cdot 3 = 9.$$

At no P_i, then, are there more than 9 mixed angles, and so the total number cannot exceed $7 \cdot 9 = 63$ for the entire configuration. Thus the number of mixed triangles cannot be more than 31.

However, the seven points determine $\binom{7}{3} = 35$ triangles altogether, and it follows, then, that *at least four* of them must be monochromatic. (For the general proof that the number of monochromatic triangles determined by n points is at least

$$\binom{n}{3} - \left[\frac{n}{2}\left[\left(\frac{n-1}{2}\right)^2\right]\right],$$

where the square brackets indicate "integer part," see the essay Chromatic Graphs, by Harold Dorwart and Dan Finkbeiner, in *Mathematical Plums*, Vol. 4, Dolciani Mathematical Expositions, MAA, 1979, pages 7–8).

Consider, then, any set of four of the monochromatic triangles. Two of them with different colors could not have a common side and would thus imply the desired conclusion. Suppose, therefore, that all four of the triangles are the same color, say red. Again, if two of these fail to have a common side, the conclusion is immediate, and so let us suppose that each pair of these triangles do have a side in common. In this case there are only two ways they could possible fit together: either

(i) they all hinge along a common side P_iP_j like propeller blades, or
(ii) they are the four faces of a tetrahedron $P_iP_jP_kP_l$.

In any event, we shall see that there still must be two monochromatic triangles somewhere in the configuration which have no side in common.

In Figure 95(i), we see immediately that, if P_1P_2 is shared by all four of the triangles, any red side of $\triangle P_3P_4P_5$ would complete an all-red triangle with two of the red edges at P_1, and that otherwise $\triangle P_3P_4P_5$ itself would be all-blue. In either case, then, there is a monochromatic triangle which does not have an edge in common with $\triangle P_1P_2P_6$.

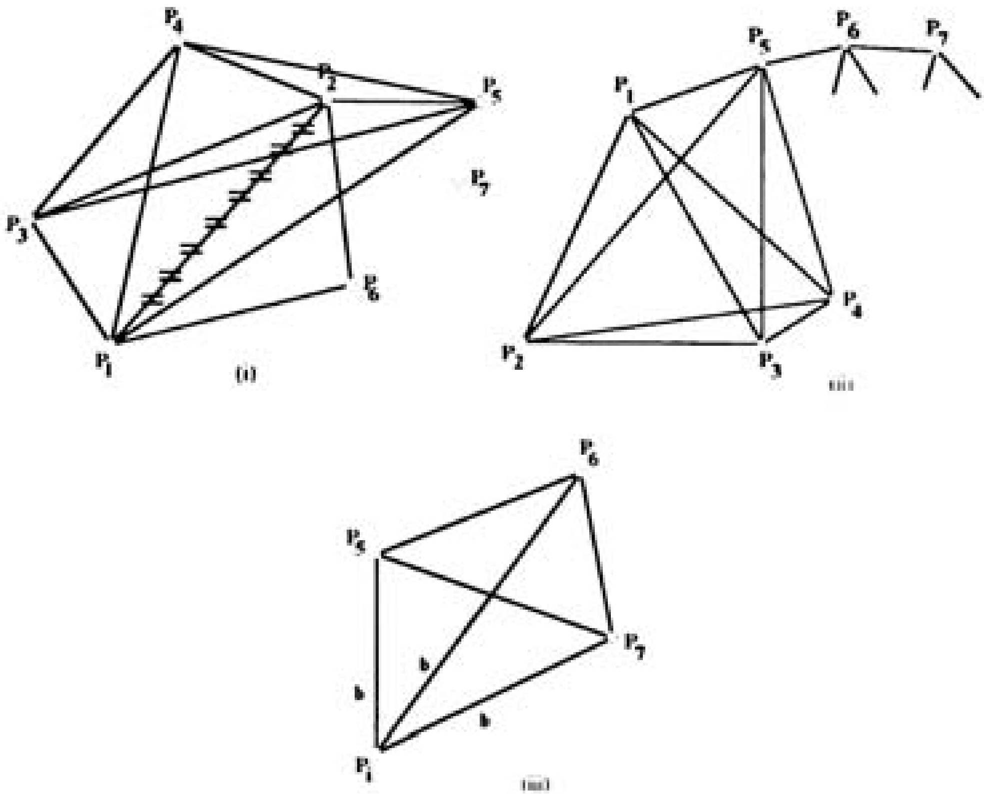


FIGURE 95(i–iii)

If the four red triangles are the faces of a tetrahedron $P_1P_2P_3P_4$ (Figure 95(ii)), we might argue as follows. If any two of the four edges from P_5 to the vertices of the tetrahedron were also red, an additional all-red triangle would result which, although having a common side with two faces of the tetrahedron, would *not* share a common edge with the other two, and the desired conclusion would follow. The same is true for the edges at P_6 and P_7 and so, if there is more than one red edge to the tetrahedron from any of P_5, P_6, and P_7, the conclusion follows.

Finally, then, suppose there is not more than one red edge to the tetrahedron from any of P_5, P_6, and P_7. In this case, at least one vertex P_i of the tetrahedron must receive a blue edge from all three of P_5, P_6, and P_7. Thus, from Figure 95(iii) it is clear that either $P_5P_6P_7$ is itself a monochromatic red triangle or a monochromatic blue triangle is formed at P_i, and the argument is complete.

From a Chinese Contest

If Paul Erdős can't solve a problem in one of his special fields, it's almost a certainty that the problem is very difficult. Unfortunately, Erdős is not always around to help you rate a problem that you can't seem to get on your own. Once a problem outruns one's own powers, there is really no telling how difficult it is. I guess there's a lot of truth in the old saying "It's easy if you can do it and hard if you can't."

I worked casually on and off for quite a while on the following problem, just a few minutes at a time whenever I thought of it. I was beginning to wonder if I would ever get it when it finally surrendered one day. Now that it's all over, I can't imagine why it took so long. Surely it wouldn't be right to call it a difficult problem, and the solution now seems to be reasonably natural and straightforward. The problem comes from a contest that was held in China in the early 1960's; I believe it was intended for some level of secondary school student.

The Problem

Prove that $[\sqrt{n}+\sqrt{n+1}]=[\sqrt{4n+1}]=[\sqrt{4n+2}]=[\sqrt{4n+3}]$ for all positive integers n, where $[x]$ denotes the greatest integer $\leq x$.

Solution

Clearly

$$n=\sqrt{n\cdot n}<\sqrt{n(n+1)}<\sqrt{(n+1)^2}<n+1,$$

and doubling gives

$$2n < 2\sqrt{n(n+1)} < 2n + 2. \tag{A}$$

Also,

$$2\sqrt{n(n+1)} = (\sqrt{n} + \sqrt{n+1})^2 - (2n+1).$$

Therefore (A) gives

$$2n < (\sqrt{n} + \sqrt{n} + 1)^2 - (2n+1) < 2n + 2$$

and, adding $2n + 1$ throughout, we have

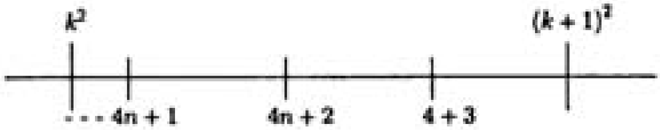
$$4n + 1 < (\sqrt{n} + \sqrt{n} + 1)^2 < 4n + 3,$$

and

$$\sqrt{(4n+1)} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+3}. \tag{B}$$

Now, $4n + 1$ is either a perfect square itself or it lies between some pair of consecutive squares; that is, for some positive integer k, we have

$$k^2 \leq 4n + 1 < (k+1)^2.$$



Now let's make our way along the integers from k^2 toward $(k+1)^2$. Before reaching $(k+1)^2$ we will have to get past $4n + 1$, if indeed we don't begin there, immediately after which we will encounter $4n + 2$, and then $4n + 3$. But we can't have reached the destination of $(k+1)^2$ yet because no perfect square can be congruent to 2 or 3 modulo 4. That is to say, all these integers must lie between k^2 and $(k+1)^2$:

$$k^2 \leq 4n + 1 < 4n + 2 < 4n + 3 < (k+1)^2,$$

from which we have

$$k \leq \sqrt{4n+1} < \sqrt{4n+2} < \sqrt{4n+3} < k + 1.$$

Since $\sqrt{n} + \sqrt{n+1}$ lies between $\sqrt{4n+1}$ and $\sqrt{4n+3}$ (by (B)), then all four of the numbers $\sqrt{4n+1}$, $\sqrt{4n+2}$, $\sqrt{4n+3}$, and $(\sqrt{n} + \sqrt{n} + 1)$, lie in the half-closed interval $[k, k+1)$ between k and $k + 1$, implying that they all have the same integral part, namely k.

A Japanese Temple Geometry Problem

With mathematical contact with the West cut off by Japan's isolationist policy, Japanese mathematics during the greater part of the Edo period (1603–1867) was strictly home-grown. As it happened, people from all walks of life, from farmer to samurai, were very keen on synthetic geometry and made many remarkable discoveries. It was the custom of the times to inscribe such gems on wooden tablets and hang them in the precincts of their shrines and temples. Unfortunately, Japan is a land of earthquakes and electrical storms, and many of these treasures have been lost in fires over the years. Nevertheless, a large remnant has been preserved and a selection of some 250 of these marvellous problems have been issued in English in a book prepared by Hidetosi Fukagawa (Yokosuka High School, Tokai-city, Aichi), the world's leading authority on these "sangaku", in collaboration with the eminent U.S. geometer Dan Pedoe (Minneapolis, Minnesota). The following problem is in their book, but the solutions given here are by contributors to *Crux Mathematicorum*, not from the book.

P1121

(*Crux Mathematicorum*, 1987, 194)

X is a point on a chord L of a given circle C, and circles A and B are drawn on opposite sides of L so as to touch L at X and be tangent to C (Figure 96). Prove that, wherever X might be chosen on L, the relative sizes of A and B are always the same, that is,

$$\frac{\text{radius of } A}{\text{radius of } B} = \frac{a}{b} = a \text{ constant} \left(= \frac{\text{radius of } D}{\text{radius of } E} \text{ at any other point } Y \right).$$

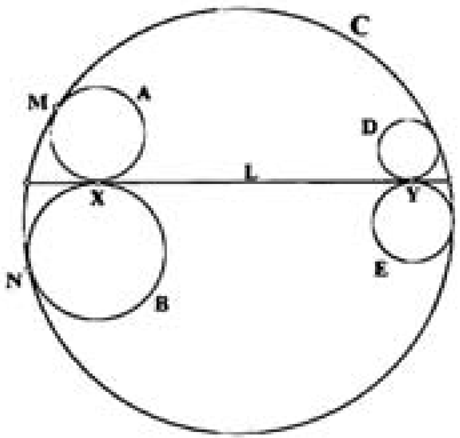


FIGURE 96

Solutions

Our first solution is a brillant piece of work by Dan Sokolowsky (Williamsburg, Virginia).

He begins by drawing PXQ perpendicular to L at X (Figure 97). Since L is a common tangent to A and B, the perpendiculars PX and XQ are diameters, and, if M and N are the points of contact, $\angle PMX$ is a right angle. Hence, extending MP and MX determines a diameter RS in C.

Now, the centers U and V of A and C are in line with their point of contact M, and it is clear that triangles MUX and MVS are isosceles. Since they share a

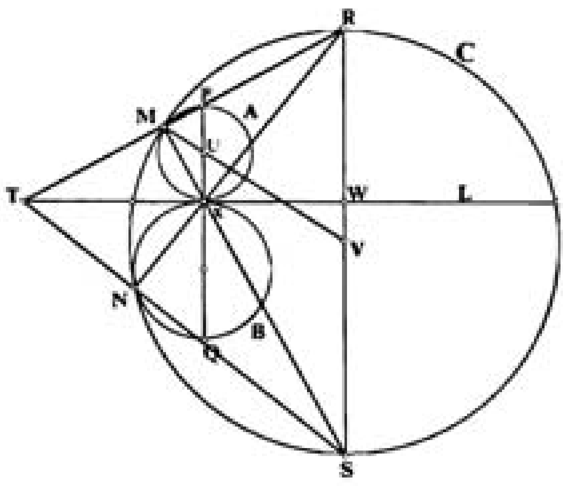


FIGURE 97

common base angle at M, the base angles at X and S are also equal, and the lines PXQ and RS are parallel. Thus RS is perpendicular to L.

In the same way, extending NX and NQ also determines the diameter of C which is perpendicular to L, and so NX and NQ also pass through R and S, respectively.

Now let RM and SN be extended to meet at some point T, completing triangle TRS; at this point, we don't know that T lies on L. In this triangle, RN and SM are clearly altitudes, making X the orthocenter, and therefore TX determines the third altitude (to RS). But L is the perpendicular to RS through X, and therefore T does indeed lie on L.

As a result, PQ is a line parallel to RS in $\triangle TRS$ (Figure 98), and we have two pairs of similar triangles

$$(TXP, TWR) \quad \text{and} \quad (TXQ, TWS).$$

Observing that PX and QX are diameters of A and B, we have

$$\frac{2a}{RW} = \frac{TX}{TW} = \frac{2b}{WS}, \qquad \text{implying} \quad \frac{a}{b} = \frac{2a}{2b} = \frac{RW}{WS}.$$

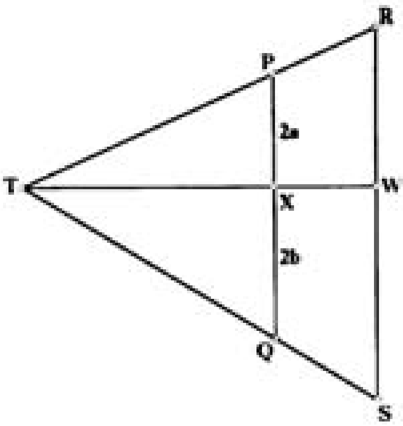


FIGURE 98

But $\frac{RW}{WS}$ is simply the ratio of the parts into which L divides the diameter of C which is perpendicular to it, and is completely *independent* of the position of X on L. The conclusion follows.

In our second solution, by Sam Baethge (San Antonio, Texas), the problem is solved as if by magic almost before we get properly started. How easily the lock turns for the right key.

Let the centers of A, B, and C be J, K, and O, and let their radii, respectively, be a, b, and R. Also, let $OW = d$ and $OZ = p$ be perpendiculars to L and the line of centers JK (Figure 99). Then, it is clear that

$$JZ = a + d, \qquad KZ = b - d,$$
$$JO = R - a, \qquad \text{and} \quad KO = R - b.$$

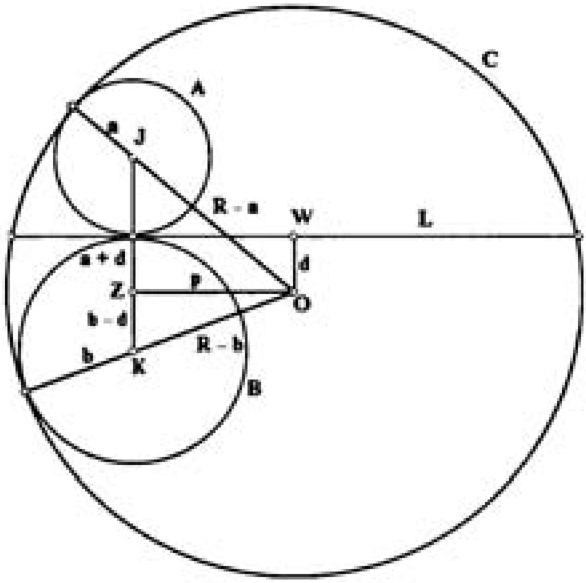


FIGURE 99

Now then, applying the Pythagorean theorem to triangles JOZ and KOZ essentially solves the problem. This gives

$$p^2 + (a + d)^2 = (R - a)^2, \tag{1}$$

and

$$p^2 + (b - d)^2 = (R - b)^2. \tag{2}$$

Solving (1), we get

$$p^2 + a^2 + 2ad + d^2 = R^2 - 2aR + a^2,$$

and

$$a = \frac{R^2 - p^2 - d^2}{2(d + R)}.$$

Similarly, (2) gives

$$p^2 + b^2 - 2bd + d^2 = R^2 - 2bR + b^2,$$

and

$$b = \frac{R^2 - p^2 - d^2}{2(R - d)}.$$

Hence

$$\frac{a}{b} = \frac{R - d}{R + d},$$

a constant.

Another very nice solution, by J. Chris Fisher (University of Regina, Saskatchewan), based on circular inversion, is also reported in *Crux Mathematicorum*.

Two Problems from the Second Balkan Olympiad, 1985

(*Crux Mathematicorum*, 1987, 71-72)

Problem 1

The matter of foreign languages is always a problem at international conferences. Of the 1985 people in attendance at a recent conference, no one spoke more than 5 languages and, in any subset of 3 of those assembled, at least 2 spoke a common language. Prove that some language was spoken by at least 200 of the people at the meeting.

Solution

(A similar solution is given in *Crux Mathematicorum*, 1991, 228.)

Perhaps the first thing to come to mind is the pigeohole principle and the possibility that the numbers involved just won't work out if no language is spoken by more than 199 people. On the basis of this assumption, then, let us try for a contradiction.

If any two of the conventioners, A and B, do *not* have a common language, then in any trio (A, B, C) to which they belong, it must be that C has a language in common with either A or B (or both). Now, by assumption there are not more than 198 others who speak a particular language spoken by A or B, and since A and B don't know more than 10 languages between them, there can't be more than $10 \cdot 198 = 1980$ people C who are eligible to complete an acceptable trio (A, B, C). This falls short of the $1985 - 2 = 1983$ others who are supposed to be able to fulfill this role, and we have a contradiction already.

It is easy to forget all about the alternative case, for the odds must be overwhelming that some two of the 1985 participants will be at linguistic loggerheads. However, we still need to consider the possibility that *every* pair (A, B) have a language in common. Fortunately, this is a trivial case, for then each of the other 1984 people must speak one of the five or fewer languages spoken by A, for an average of at least 397 (counting A).

Problem 2

(Two alternative solutions are given in *Crux Mathematicorum*, 1991, 105 and 228.)

O is the circumcenter of triangle ABC and CD is the median to AB. E is the centroid of triangle ACD. Prove that OE is perpendicular to CD if and only if $\triangle ABC$ is isosceles with $AB = AC$.

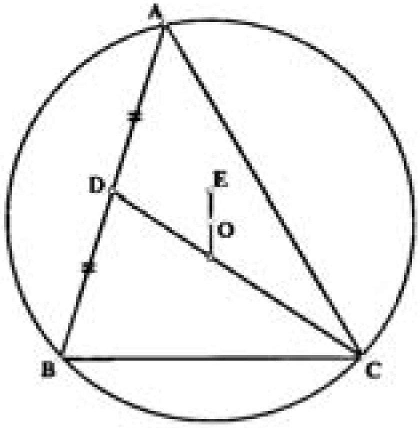


FIGURE 100

Being particularly fond of Euclidean geometry, my first attempt at any problem that is posed in Euclidean terms is always by synthetic means. After spending the better part of an afternoon puzzling over this problem, I finally decided to give the analytic approach a try; after all, perpendicular lines are very simply characterized analytically by negative-reciprocal slopes. Ten minutes later it was all over, and I was again properly chastened by the power of analytic geometry.

On further thought, it appeared that the problem might also be amenable to a solution by vectors, and a vector solution, having about the same complexity as the analytic solution, soon turned up. Since vectors seem to absorb the essential features of a configuration so effortlessly, ofter simply by adding one's way around a polygon or two, I hope you will enjoy both these solutions.

The Analytical Approach

Let the Cartesian coordinates of the vertices be $B(0,0)$, $C(6a,0)$, and $A(4b,2c)$. Then the midpoint D of AB is $(2b,c)$.

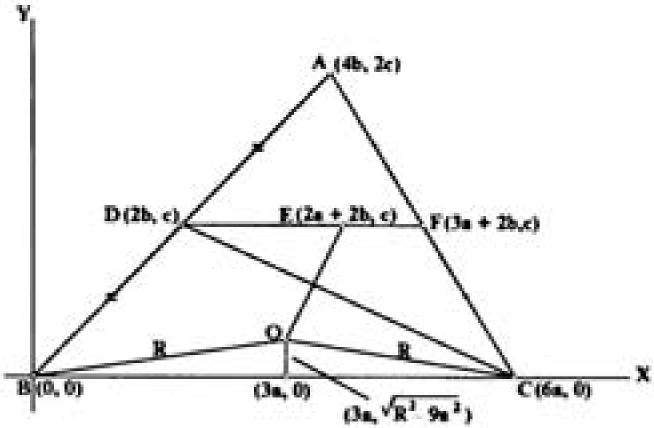


FIGURE 101

Since E is the centroid of $\triangle ADC$, DE is a median and therefore meets the opposite side AC in its midpoints $F(3a+2b,c)$.

Because each median is trisected by the centroid, the coordinates of E are $(2b+2a,c)$.

Lying on the perpendicular bisector of BC, O is easily seen to be the point $(3a,\sqrt{R^2-9a^2})$, where $R = BO$ is the circumradius of $\triangle ABC$.

Hence

$$\text{the slope of } CD = \frac{-c}{6a-2b},$$

$$\text{the slope of } OE = \frac{\sqrt{R^2-9a^2}-c}{a-2b},$$

and CD and OE are perpendicular if and only if

$$\frac{-c}{6a-2b}\cdot\frac{\sqrt{R^2-9a^2}-c}{a-2b} = -1,$$

$$c\left[\sqrt{R^2-9a^2}-c\right] = (6a-2b)(a-2b)$$

$$c\sqrt{R^2-9a^2}-c^2 = 6a^2-14ab+4b^2$$

$$c\sqrt{R^2-9a^2} = c^2+6a^2-14ab+4b^2.$$

Now, $AO = R$ gives another expression for $c\sqrt{R^2 - 9a^2}$ as follows.

$$AO^2 = (3a - 4b)^2 + (\sqrt{R^2 - 9a^2} - 2c)^2 = R^2,$$

$$(3a - 4b)^2 + R^2 - 9a^2 - 4c\sqrt{R^2 - 9a^2} + 4c^2 = R^2,$$

$$9a^2 - 24ab + 16b^2 - 9a^2 + 4c^2 = 4c\sqrt{R^2 - 9a^2},$$

and finally

$$-6ab + 4b^2 + c^2 = c\sqrt{R^2 - 9a^2}.$$

Therefore CD and OE are perpendicular if and only if

$$-6ab + 4b^2 + c^2 = c^2 + 6a^2 - 14ab + 4b^2,$$

$$8ab = 6a^2,$$

$$4b = 3a,$$

that is iff $A(4b, 2c)$ is the point $A(3a, 2c)$ on the perpendicular bisector of BC,

i.e., iff $AB = AC$.

The Vector Solution

Again, we need to note a few easy preliminaries:

(i) E trisects median DF of $\triangle ADC$, giving $\overrightarrow{DE} = \frac{2}{3}\overrightarrow{DF}$;
(ii) Since D and F are the midpoints of AB and AC,

$$\overrightarrow{DF} = \frac{1}{2}\overrightarrow{BC}, \qquad \text{giving} \quad \overrightarrow{DE} = \frac{2}{3}\overrightarrow{DF} = \frac{1}{3}\overrightarrow{BC}.$$

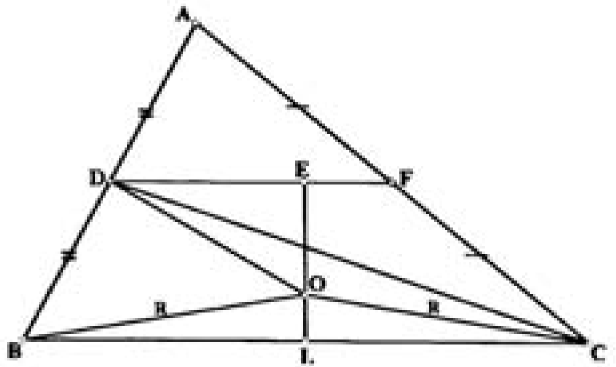


FIGURE 102

(iii) Since O lies on the perpendicular bisectors of AB and BC,

$$R\cos \measuredangle ABO = BD = \frac{1}{2}c, \quad \text{and} \quad R\cos \measuredangle OBC = BL = \frac{1}{2}a, \ldots \qquad (X)$$

using the usual notation $BC = a, AC = b$, and $AB = c$.
Now, it is clear that

$$\overrightarrow{BO} + \overrightarrow{OE} = \overrightarrow{BE} = \overrightarrow{BD} + \overrightarrow{DE},$$

giving

$$\begin{aligned}\overrightarrow{OE} &= \overrightarrow{BD} + \overrightarrow{DE} - \overrightarrow{BO}\\ &= \frac{1}{2}\overrightarrow{BA} + \frac{1}{3}\overrightarrow{BC} - \overrightarrow{BO},\end{aligned}$$

and also that

$$\overrightarrow{BD} + \overrightarrow{DC} = \overrightarrow{BC},$$

giving

$$\overrightarrow{DC} = \overrightarrow{BC} - \overrightarrow{BD} = \overrightarrow{BC} - \frac{1}{2}\overrightarrow{BA}.$$

Recalling that $\vec{u} \cdot \vec{v} = |u|\,|v| \cos\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$, the dot product of $\overrightarrow{OE}$ and $\overrightarrow{DC}$ is

$$\begin{aligned}\overrightarrow{OE} \cdot \overrightarrow{DC} &= \left(\frac{1}{2}\overrightarrow{BA} + \frac{1}{3}\overrightarrow{BC} - \overrightarrow{BO}\right)\cdot\left(\overrightarrow{BC} - \frac{1}{2}\overrightarrow{BA}\right)\\ &= \frac{1}{2}\overrightarrow{BA}\cdot\overrightarrow{BC} - \frac{1}{4}\overrightarrow{BA}\cdot\overrightarrow{BA} + \frac{1}{3}\overrightarrow{BC}\cdot\overrightarrow{BC}\\ &\quad - \frac{1}{6}\overrightarrow{BC}\cdot\overrightarrow{BA} - \overrightarrow{BO}\cdot\overrightarrow{BC} + \frac{1}{2}\overrightarrow{BO}\cdot\overrightarrow{BA}\\ &= \frac{1}{2}ca\cos B - \frac{1}{4}c^2 + \frac{1}{3}a^2 - \frac{1}{6}ac\cos B\\ &\quad - Ra\cos \measuredangle OBC + \frac{1}{2}Rc\cos \measuredangle ABO\\ &= \frac{1}{3}ca\cos B - \frac{1}{4}c^2 + \frac{1}{3}a^2 - \frac{1}{2}a^2 + \frac{1}{4}c^2\\ &\quad \text{(by the results of } (X)\text{, above)}\\ &= \frac{1}{3}ca\cos B - \frac{1}{6}a^2.\end{aligned}$$

Finally, from the law of cosines, we have $ca\cos B = \dfrac{a^2+c^2-b^2}{2}$, giving

$$\begin{aligned}\overrightarrow{OE}\cdot\overrightarrow{DC} &= \frac{a^2+c^2-b^2}{6} - \frac{1}{6}a^2\\ &= \frac{1}{6}(c^2-b^2),\end{aligned}$$

which is obviously equal to zero, implying the desired perpendicularity, if and only if $c = b$, i.e., if and only if

$$AB = AC.$$

A Property of Pedal Triangles

(P 1076, *Crux Mathematicorum*, 1987, 62)

The triangle DEF determined by dropping perpendiculars to the sides of a triangle ABC from a point P inside it is called the *pedal* triangle of P relative to ABC. In this problem we are asked to establish an engaging formula connecting the area of a triangle with its pedal triangles.

If x, y, and z are the distances from P to the vertices, A, B, and C, respectively, prove that, for *every choice of the point* P inside $\triangle ABC$,

$$x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C + 8\triangle DEF = 4\triangle ABC.$$

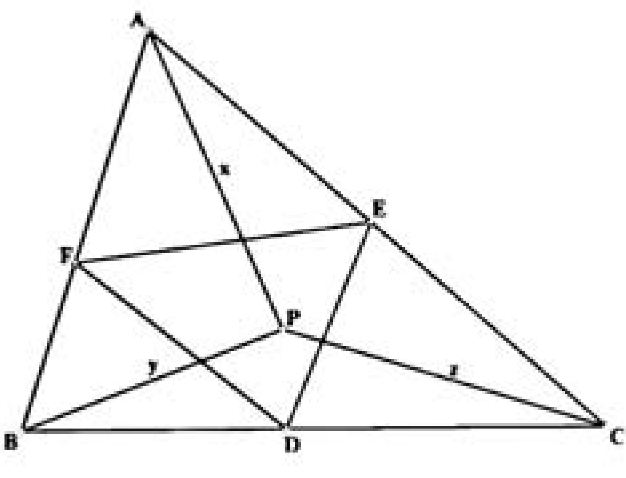


FIGURE 103

This is the discovery of the ingenious creator and solver of problems Murray Klamkin (University of Alberta, retired), and the following beautiful solution is by the eminent Greek geometer George Tsintsifas (Thessaloniki).

Solution

Since PE and PF are perpendicular to sides of the triangle, the quadrilateral $AFPE$ is cyclic and, in fact, AP is a diameter of the circumcircle, making its midpoint L the center and $\frac{1}{2}AP = \frac{1}{2}x$ the radius (Figure 104(i)). Hence each radius LF and LE is $\frac{1}{2}x$, and the angle which is subtended by the chord EF at the center L is double the angle it subtends at A on the circumference, making $\measuredangle FLE = 2A$.

Since $\triangle FLE$ is isosceles, the altitude LT bisects both the vertical angle at L and the base FE, and we have (see the small figure in Figure 104(i))

$$\measuredangle FLT = A, \quad \text{and} \quad FT = \frac{1}{2}FE.$$

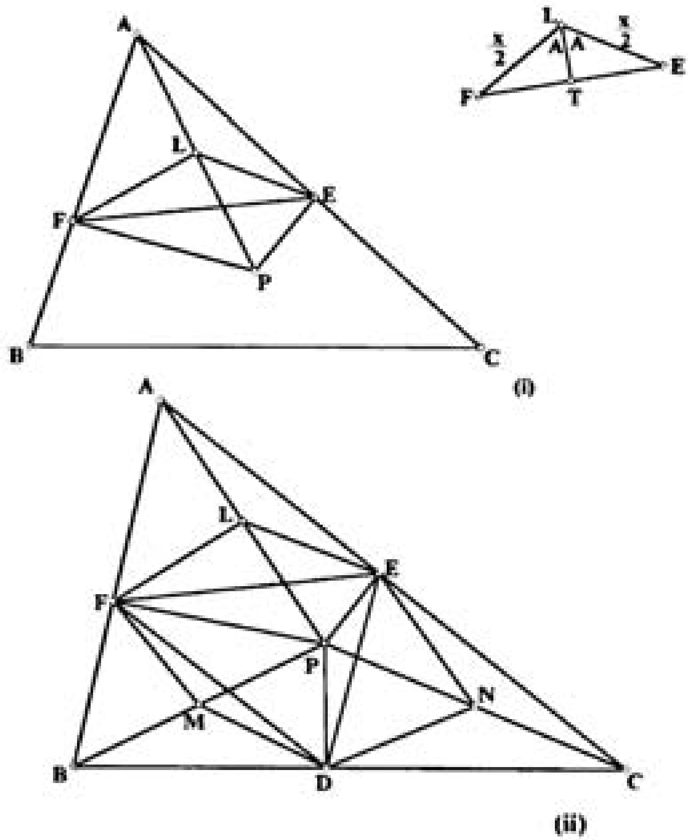


FIGURE 104(i–ii)

But from $\triangle FLT$, we get $FT = \frac{x}{2}\sin A$.
Thus

$$FT = \frac{1}{2}FE = \frac{1}{2}x\sin A,$$

giving

$$EF = x\sin A.$$

Also,

$$LT = \frac{x}{2}\cos A, \quad \text{and so} \quad x\cos A = 2LT.$$

Consequently,

$$\begin{aligned} x^2\sin 2A &= x^2(2\sin A\cos A) \\ &= 2(x\sin A)(x\cos A) \\ &= 2EF\cdot 2LT \\ &= 8\left(\frac{1}{2}EF\cdot LT\right), \end{aligned}$$

and we get

$$x^2\sin 2A = 8\triangle FLE.$$

Similarly, in Figure 104(ii),

$$y^2\sin 2B = 8\triangle FMD,$$

and

$$z^2\sin 2C = 8\triangle DNE.$$

Thus we have

$$x^2\sin 2A + y^2\sin 2B + z^2\sin 2C = 8(\triangle FLE + \triangle FMD + \triangle DNE).$$

Adding 8 times the pedal triangle DEF, we get

$$x^2\sin 2A + y^2\sin 2B + z^2\sin 2C + 8\triangle DEF = 8\,(\text{hexagon } FLENDM).$$

Finally, since L, M, and N are the midpoints of AP, BP, and CP, it is easy to see that in $\triangle ABC$ there is as much area outside the hexagon $FLENDM$ as there is inside it:

median FL bisects $\triangle APF$, and similarly for the other medians LE, EN, ND, DM, and MF.

Therefore

$$\triangle ABC = 2(FLENDM),$$

giving

$$4(\triangle ABC) = 8(FLENDM),$$

from which the desired formula follows immediately.

We note that if $\triangle ABC$ is obtuse-angled, say at A, then $\sin 2A$ would be negative, making the sign of $x^2 \sin 2A$ $(= 8\triangle FLE)$ negative, obliging us to consider the area of $\triangle FLE$ to be negative. But in the case of an obtuse angle at A, Figure 105 shows that the area of $\triangle FLE$ needs to be subtracted from the pedal triangle DEF in order to add the right amount to triangles MFD and DNE to obtain the hexagon $FLENDM$:

$$FLENDM = \triangle MFD + \triangle DNE + (\triangle DEF - \triangle FLE).$$

Thus the negative area validates the above argument in the case of obtuse triangles, and we conclude that the equation under investigation holds for all triangles ABC.

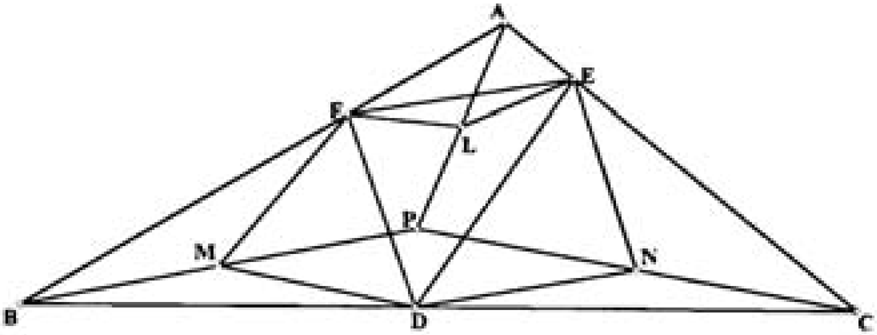


FIGURE 105

Three More Solutions by George Evagelopoulos

The three problems in this section are somewhat technical and more difficult and they require a little higher level of concentration. Still, they are very engaging problems and George's perceptive solutions amply reward the extra effort.

Problem 1

A challenging inequality from Russia. ([1985,73], [1987,491])

Suppose $x_1, x_2, \ldots, x_n$ are any n positive real numbers ≤ 1, not necessarily distinct, listed in *non-increasing* order of magnitude:

$$1 \geq x_1 \geq x_2 \geq \cdots \geq x_n > 0.$$

Then, if a is any value in $[0,1]$, prove the rather formidable-looking inequality

$$(1 + x_1 + \cdots + x_n)^a \leq 1 + x_1^a + \frac{1}{2}(2x_2)^a + \frac{1}{3}(3x_3)^a + \cdots + \frac{1}{n}(nx_n)^a.$$

Solution

It comes as no surprise that problems like this are often amenable to induction. However, just how to carry through the details may be a very complicated undertaking.

We can see quickly that the inequality holds for a single number x_1. Since, $x_1 > 0$, we have $1 + x_1 > 1$, and because $0 \leq a \leq 1$, than ath power of $1 + x_1$

would tend to be smaller than $1 + x_1$:

$$(1 + x_1)^a \leq 1 + x_1.$$

On the other hand, since $x_1 \leq 1$ and $0 \leq a \leq 1$, then $x_1{}^a$ would tend to be greater than x_1, and we have altogether that

$$(1 + x_1)^a \leq 1 + x_1 \leq 1 + x_1{}^a,$$

satisfying the given inequality.

Now suppose the inequality holds for some positive integer $k \geq 1$, i.e.,

$$(1 + x_1 + \cdots + x_k)^a \leq 1 + x_1{}^a + \frac{1}{2}(2x_2)^a + \cdots + \frac{1}{k}(kx_k)^a,$$

and that $\{x_1, x_2, \ldots, x_{k+1}\}$ is an appropriate set of $k + 1$ numbers.

If we could show that the difference

$$(1 + x_1 + \cdots + x_{k+1})^a - (1 + x_1 + \cdots + x_k)^a \leq \frac{1}{k+1}[(k + 1)x_{k+1}]^a, \quad \text{(A)}$$

then the desired conclusion would follow by induction, for then

$$\begin{aligned}(1 + x_1 + \cdots + x_{k+1})^a &\leq (1 + x_1 + \cdots + x_k)^a + \frac{1}{k+1}[(k + 1)x_{k+1}]^a \\ &\leq 1 + x_1 + \tfrac{1}{2}(2x_2)^a + \cdots + \frac{1}{k}(kx_k)^a + \frac{1}{k+1}[(k+1)x_{k+1}]^a.\end{aligned}$$

To establish (A), George considers the left side and begins by removing a factor equal to the second term itself, to get the expression

$$(1 + x_1 + \cdots + x_k)^a\left[\left(1 + \frac{x_{k+1}}{1 + x_1 + \cdots + x_k}\right)^a - 1\right].$$

Since $0 \leq a \leq 1$, then

$\left(1 + \dfrac{x_{k+1}}{1 + x_1 + \cdots + x_k}\right)^a$ tends to be smaller than $\left(1 + \dfrac{x_{k+1}}{1 + x_1 + \cdots + x_k}\right)^1$, and we obtain

$$\begin{aligned}&(1 + x_1 + \cdots + x_{k+1})^a - (1 + x_1 + \cdots + x_k)^a \\ &= (1 + x_1 + \cdots + x_k)^a\left[\left(1 + \frac{x_{k+1}}{1 + x_1 + \cdots + x_k}\right)^a - 1\right] \\ &\leq (1 + x_1 + \cdots + x_k)^a\left[\left(1 + \frac{x_{k+1}}{1 + x_1 + \cdots + x_k}\right)^1 - 1\right] \\ &= (1 + x_1 + \cdots + x_k)^{a-1}x_{k+1}. \qquad \text{(B)}\end{aligned}$$

Next, because of the non-increasing order of the x_i, we have

$$\begin{aligned} 1 + x_1 + \cdots + x_k &\geq x_{k+1} + x_{k+1} + \cdots + x_{k+1} \quad (k+1 \text{ times}) \\ &= (k+1)x_{k+1}. \end{aligned}$$

Since $1 + x_1 + \cdots + x_k$ is bigger than 1, and also $1 - a \geq 0$, this yields

$$(1 + x_1 + \cdots + x_k)^{1-a} \geq \left[(k+1)x_{k+1}\right)^{1-a}.$$

Negating the exponents inverts these quantities and reverses the inequality to give

$$(1 + x_1 + \cdots + x_k)^{a-1} \leq [(k+1)x_{k+1}]^{a-1},$$

and multiplying by x_{k+1}, we get

$$\begin{aligned} (1 + x_1 + \cdots + x_k)^{a-1}x_{k+1} &\leq [(k+1)x_{k+1}]^{a-1}x_{k+1} \\ &= \frac{1}{k+1}[(k+1)x_{k+1}]^a. \end{aligned} \tag{C}$$

Finally, then, combining results (B) and (C), we have

$$\begin{aligned} &(1 + x_1 + \cdots + x_{k+1})^a - (1 + x_1 + \cdots + x_k)^a \\ &\leq (1 + x_1 + \cdots + x_k)^{a-1}x_{k+1} && \text{(by (B))} \\ &\leq \frac{1}{k+1}[(k+1)x_{k+1}]^a && \text{(by (C))}, \end{aligned}$$

as desired.

Problem 2

Next we turn to the final round of the 1985 Bulgarian olympiad [1988, 230].

Let α_a be the greatest *odd* divisor of the positive integer a, and let S_b denote the sum

$$S_b = \frac{\alpha_1}{1} + \frac{\alpha_2}{2} + \cdots + \frac{\alpha_b}{b} = \sum_{a=1}^{b} \frac{\alpha_a}{a}.$$

Prove that the sequence $\left\{\frac{S_b}{b}\right\}$ converges and find its limit.

Solution

Clearly the greatest odd divisor α_a of an integer a is what's left after it has been divided by 2 as often as possible:

$$a = 2^{r_a} \cdot \alpha_a,$$

where 2^{r_a} is the greatest power of 2 that divides a; e.g., $\alpha_{68} = 17$ (since $68 = 2^2 \cdot 17$), and $\alpha_{69} = 69$ ($69 = 2^0 \cdot 69$). Consequently $\frac{\alpha_a}{a} = 1/2^{r_a}$, and S_b is simply

$$S_b = \frac{1}{2^{r_1}} + \frac{1}{2^{r_2}} + \frac{1}{2^{r_3}} + \cdots + \frac{1}{2^{r_b}}.$$

Now, lots of integers have the same greatest power of 2 in their makeup; for example, the greatest power of 2 that divides each of 12, 28, and 52 is $2^2 = 4$. We would be able to evaluate the series for S_b, then, if we could determine *how many* of its terms were equal to $1/2^0$, how many were equal to $1/2^1$, and so on. As we have noted, the term $1/2^k$ occurs for each integer a in the range $\{1, 2, \ldots, b\}$ for which 2^k is the greatest power of 2 that divides it. Happily, we can count these integers as follows.

Every 2^kth integer is divisible by 2^k, and the number of multiples of 2^k which are less than or equal to b is $[b/2^k]$, the *integer part* of $b/2^k$. Since we are interested only in the integers for which 2^k is the *greatest* power of 2 that divides them, we do not want to include in our count any integer that is divisible by 2^{k+1}. But every second multiple of 2^k is (an even number) $\cdot 2^k$, i.e., $2t \cdot 2^k$, making it a multiple of 2^{k+1}. Since the number of unwanted multiples of 2^{k+1} in the range $\{1, 2, \ldots, b\}$ is $[b/2^{k+1}]$, the number of times 2^k occurs as the greatest power of 2 is precisely

$$\left[\frac{b}{2^k}\right] - \left[\frac{b}{2^{k+1}}\right].$$

The contribution of the corresponding terms toward the sum S_b is

$$\left\{\left[\frac{b}{2^k}\right] - \left[\frac{b}{2^{k+1}}\right]\right\} \cdot \frac{1}{2^k},$$

and the entire sum is given by

$$S_b = \sum_{k \geq 0} \{$$

a finite sum which continues until 2^k exceeds b, beyond which all the frequencies $[b/2^k] = 0$.

Expanding and simplifying a few terms in this sum reveals a useful alternative formula for S_b:

$$S_b = \left\{\left[\frac{b}{2^0}\right] - \right.$$

and we have

$$S_b = b - \sum_{k\geq 1} \frac{1}{2^k}\left[\frac{b}{2^k}\right].$$

Now, by the definition of $[x]$, it is clear that $[x]$, while not exceeding x itself, cannot be as small as $x - 1$:

$$x - 1 < [x] \leq x.$$

Hence, for all $k \geq 1$,

$$\frac{b}{2^k} - 1 < \left[\frac{b}{2^k}\right] \leq \frac{b}{2^k},$$

which, multiplying through by $1/2^k$, gives

$$\frac{b}{4^k} - \frac{1}{2^k} < \frac{1}{2^k}\left[\frac{b}{2^k}\right] \leq \frac{b}{4^k},$$

and, summing over all $k \geq 1$, we get

$$b\sum_{k\geq 1}\frac{1}{4^k} - \sum_{k\geq 1}\frac{1}{2^k} < \sum_{k\geq 1}\frac{1}{2^k}\left[\frac{b}{2^k}\right] \leq b\sum_{k\geq 1}\frac{1}{4^k}.$$

Evaluating these geometric series, we get

$$\sum_{k\geq 1}\frac{1}{4^k} = \frac{1/4}{1-1/4} = \frac{1}{3},$$

and

$$\sum_{k\geq 1}\frac{1}{2^k} = \frac{1/2}{1-1/2} = 1,$$

and so

$$\frac{b}{3} - 1 < \sum_{k\geq 1}\frac{1}{2^k}\left[\frac{b}{2k}\right] \leq \frac{b}{3}.$$

Multiplying through by -1 reverses these inequalities, giving

$$-\frac{b}{3} \le -\sum_{k\ge 1} \frac{1}{2^k}\left[\frac{b}{2^k}\right] < 1 - \frac{b}{3},$$

and, adding b throughout, yields

$$\frac{2}{3}b \le S_b < 1 + \frac{2}{3}b.$$

Finally, then, dividing by b, we get

$$\frac{2}{3} \le \frac{S_b}{b} < \frac{2}{3} + \frac{1}{b},$$

and

$$\lim_{b\to\infty} \frac{S_b}{b} = \frac{2}{3}.$$

Problem 3

Finally, let us consider a problem from the 1984 All-Union Russian olympiad [1986, 235].

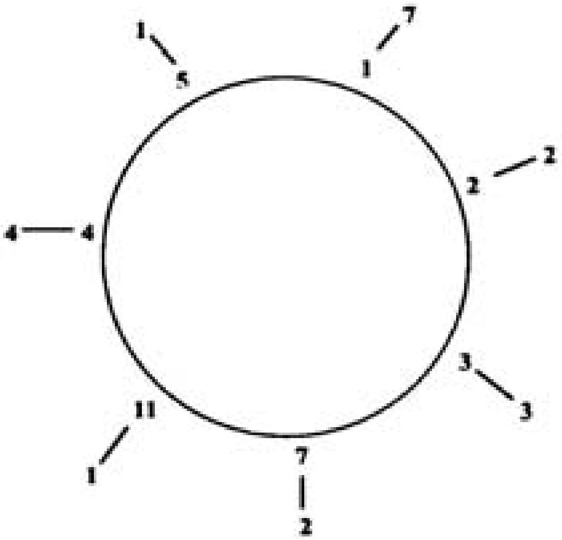


FIGURE 106

Suppose $x_1, x_2, \ldots, x_n$ are four or more positive integers, arranged in order around a circle, so that the sum of the neighbors of each x_i is a multiple of x_i itself, that is,

$$\frac{x_{i-1} + x_{i+1}}{x_i} = k_i, \quad \text{a positive integer (where } x_{n+1} = x_1\text{)}.$$

Prove that the sum S_n of all these integers k_i is always at least $2n$ but never as great as $3n$:

$$2n \leq S_n < 3n.$$

An example with $n = 7$ (Figure 106):

$$S_7 = 7 + 2 + 3 + 2 + 1 + 4 + 1 = 20$$

and

$$14 \leq S_7 = 20 < 21.$$

Solution

Clearly

$$\begin{aligned} S_n &= k_1 + k_2 + \cdots + k_n \\ &= \frac{x_n + x_2}{x_1} + \frac{x_1 + x_3}{x_2} + \frac{x_2 + x_4}{x_3} + \cdots + \frac{x_{n-1} + x_1}{x_n}, \end{aligned}$$

containing both fractions $\frac{x_{i+1}}{x_i}$ and $\frac{x_i}{x_{i+1}}$ for each i (recall $x_{n+1} = x_1$). Since $\frac{a}{b} + \frac{b}{a}$ is always at least 2 for positive real numbers a and b, we quickly have

$$S_n = \sum_{i=1}^{n} \left(\frac{x_{i+1}}{x_i} + \frac{x_i}{x_{i+1}} \right) \geq 2n.$$

Now let us turn to the more challenging proposition that $S_n < 3n$.

Observing that the claim is valid for just three integers, George wisely begins his inductive proof with the case $n = 3$ rather than the more detailed case $n = 4$.

(a) $n = 3$:

$$S_3 = \frac{x_3 + x_2}{x_1} + \frac{x_1 + x_3}{x_2} + \frac{x_2 + x_1}{x_3}.$$

It is possible, of course, that the numbers x_i are all the same. In this event, each $k_i = 2$, and we easily get

$$S_3 = 2 + 2 + 2 = 6 < 3 \cdot 3 = 3n.$$

Suppose, then, that some two of x_1, x_2, x_3 are unequal, and that they are labelled so that the biggest is x_1 and the smallest is x_3. Neither the biggest nor the smallest need be unique, and so there are actually three possibilities for x_2:

$$x_1 > x_2 > x_3, \quad x_1 > x_2 = x_3, \quad x_1 = x_2 > x_3.$$

In any case, we can always count on the two relations

$$x_1 \geq x_2 \quad \text{and} \quad x_1 > x_3.$$

Consequently,

$$2x_1 = x_1 + x_1 > x_2 + x_3,$$

implying that

$$k_1 = \frac{x_2 + x_3}{x_1} < 2.$$

Since k_1 is a positive integer, it must be that $k_1 = 1$, in which case it follows that $x_1 = x_2 + x_3$. Substituting this for x_1 in the expression for S_3, we have

$$\begin{aligned} S_3 = k_1 + k_2 + k_3 &= 1 + \frac{x_1 + x_3}{x_2} + \frac{x_1 + x_2}{x_3} \\ &= 1 + \frac{x_2 + 2x_3}{x_2} + \frac{2x_2 + x_3}{x_3} \\ &= 1 + \left(1 + \frac{2x_3}{x_2}\right) + \left(\frac{2x_2}{x_3} + 1\right). \end{aligned}$$

Because k_2 and k_3 are integers, then so are the fractions in this final expression, and since they are clearly positive, they are, in fact, positive integers. But their product $\left(\frac{2x_3}{x_2}\right)\left(\frac{2x_2}{x_3}\right)$ is 4, implying that their values must either be (2 and 2) or (1 and 4). In any case, then, their sum cannot exceed 5, we have

$$S_3 \leq 1 + 1 + 1 + 5 = 8 < 3 \cdot 3 = 3n.$$

Thus the claim is valid for $n = 3$.

(b) Now suppose that $S_{n-1} < 3(n-1)$ for some $n - 1 \geq 3$ and that $\{x_1, x_2, \ldots, x_n\}$ is an appropriate set of n integers.

It turns out that the above analysis also works in this general case. If all the x_i are equal, we immediately have

$$S_n = 2n < 3n,$$

and so let us suppose that there are at least two different values among the x_i.

Again we focus on an integer x_i of maximum value. Of course, both the neighbors of a maximum x_i might also be this same maximum value. However, since there are at least two different values among the x_i, some maximum integer x_i must have at least one smaller neighbor. Let x_n denote such a maximum integer. Then we can argue again as above:

$$2x_n = x_n + x_n > x_{n-1} + x_1, \Rightarrow k_n = \frac{x_{n-1} + x_1}{x_n} < 2,$$

making $k_n = 1$ and $x_n = x_{n-1} + x_1$.

Accordingly, the integer

$$k_1 = \frac{x_n + x_2}{x_1} = \frac{x_{n-1} + x_1 + x_2}{x_1} = 1 + \frac{x_{n-1} + x_2}{x_1},$$

making the fraction

$$r_1 = \frac{x_{n-1} + x_2}{x_1} \quad \text{a positive integer;}$$

similarly,

$$k_{n-1} = \frac{x_{n-2} + x_n}{x_{n-1}} = \frac{x_{n-2} + x_{n-1} + x_1}{x_{n-1}} = 1 + \frac{x_{n-2} + x_1}{x_{n-1}},$$

making $s = \frac{x_{n-2} + x_1}{x_{n-1}}$ a positive integer. Summarizing, we have $k_1 = 1 + r$ and $k_{n-1} = 1 + s$, where r and s are positive integers.

Now consider the set $\{x_1, x_2, \ldots, x_{n-1}\}$ obtained by deleting this maximum integer x_n. The question arises whether the induction hypothesis can be applied to this reduced set, that is, whether all the associated fractions $k'_1, k'_2, \ldots, k'_{n-1}$ are positive integers. Since the values of $k'_2, k'_3, \ldots, k'_{n-2}$ are unchanged (that is, $k'_2 = k_2, k'_3 = k_3, \ldots, k'_{n-2} = k_{n-2}$), everything depends on k'_1 and k'_{n-1}. But

$$k'_1 = \frac{x_{n-1} + x_2}{x_1} \;(= \text{the integer } r), \quad \text{and} \quad k'_{n-1} = \frac{x_{n-2} + x_1}{x_{n-1}} \;(= s),$$

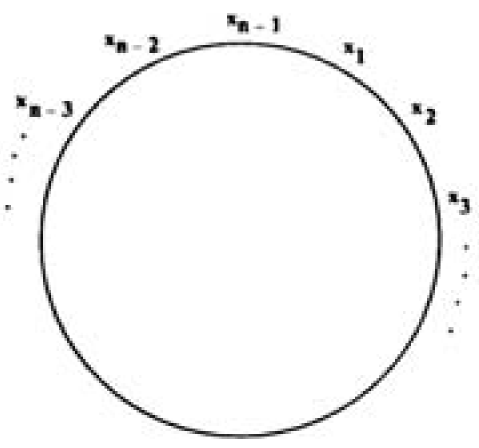


FIGURE 107

and so the induction hypothesis does indeed apply to this reduced set, and we

have

$$S_{n-1} = r + k_2 + k_3 + \cdots + k_{n-2} + s \; < \; 3(n-1).$$

Recalling that, for the full set $\{x_1, x_2, \ldots, x_n\}$, we have $k_1 = 1 + r$ and $k_{n-1} = 1 + s$, then

$$\begin{aligned} S_n &= k_1 + k_2 + \cdots + k_{n-1} + k_n \\ &= (1 + r) + k_2 + \cdots + k_{n-2} + (1 + s) + k_n \\ &= 1 + [r + k_2 + \cdots + k_{n-2} + s] + 1 + 1 \quad \text{(recall } k_n = 1\text{)} \\ &= 1 + S_{n-1} + 1 + 1 \\ &= S_{n-1} + 3 \\ &< 3(n-1) + 3 \\ &= 3n, \qquad \text{as desired.} \end{aligned}$$

The Power Mean Inequality

Suppose you would like to prove that the value of the fraction

$$\frac{(a+2b+3c)^2}{a^2+2b^2+3c^2}$$

is never greater than 6 for any positive real numbers a, b, and c, with equality if and only if $a = b = c$. The ideal tool for this is an inequality called the generalized power mean inequality. It asserts, for $t < s$, that

$$\left(\frac{p_1 a_1^t + p_2 a_2^t + \cdots + p_n a_n^t}{p_1 + p_2 + \cdots + p_n}\right)^{1/t} \le \left(\frac{p_1 a_1^s + p_2 a_2^s + \cdots + p_n a_n^s}{p_1 + p_2 + \cdots + p_n}\right)^{1/s},$$

where the p's and a's are arbitrary *positive* real numbers, with equality if and only if all the a's are equal.

In the case at hand, then, for $t = 1, s = 2$, and $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, we get

$$\left(\frac{a+2b+3c}{6}\right)^1 \le \left(\frac{a^2+2b^2+3c^2}{6}\right)^{1/2};$$

squaring gives

$$\frac{(a+2b+3c)^2}{36} \le \frac{a^2+2b^2+3c^2}{6},$$

and the desired

$$\frac{(a+2b+3c)^2}{a^2+2b^2+3c^2} \le 6$$

follows immediately.

Thus there are times when this inequality is precisely the right tool to resolve a difficult situation. Unfortunately, it is not discussed in many places and therefore I hope the following leisurely proof might be of some general interest. Although the expressions might appear to be getting out of hand sometimes, they are always easily managed, and only freshman mathematics is involved.

Essentially the inequality asserts that the function

$$f(t) = \left(\frac{p_1a_1{}^t + p_2a_2{}^t + \cdots + p_na_n{}^t}{p_1 + p_2 + \cdots + p_n}\right)^{1/t}$$

is strictly increasing with t, provided only that the a's are not all equal. (Clearly, if the a's *are* all equal to k, then $f(t) = k$ for *all* t, and is not increasing.) Therefore, let us assume that the a's are not all equal and let it be noted that the inequality is asserted for all real numbers t, not just positive values. Thus, before we can get on with the proof, we have to decide on the value of the function at $t = 0$. As we shall see, it is not difficult to show that $f(t)$ approaches the same limit L as t approaches 0 from below as it does as t approaches 0 from above, and so $f(0) = \lim f(t)$ as $t \to 0$ is the obvious choice and makes the function continuous at $t = 0$. On the strength of this, if we can show that $f(t)$ is strictly increasing for $t < 0$ and also for $t > 0$, it follows that $f(t)$ is strictly increasing everywhere.

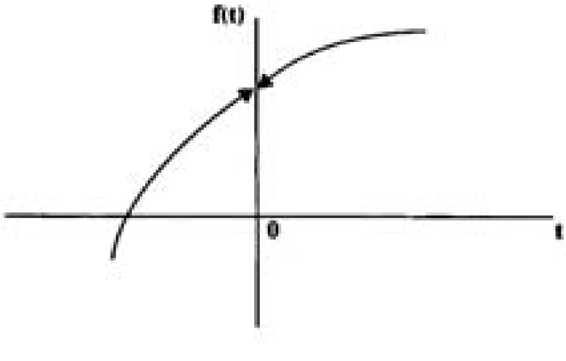


FIGURE 108

Clearly, if $\log f(t) \to k$, then $f(t) \to$ (the antilog of k). To determine $\lim f(t)$, then, let us deal with $\lim [\log f(t)]$. Denoting $p_1 + p_2 + \cdots + p_n$ by P_n, and recalling the formulas

$$\frac{d[\log f(x)]}{dx} = \frac{f'(x)}{f(x)} \quad \text{and} \quad \frac{d(a^x)}{dx} = a^x \cdot \log a,$$

L'Hôpital's rule easily yields

$$\lim_{t\to 0}[\log f(t)] = \lim_{t\to 0}\left[\frac{\log \dfrac{p_1a_1^t + \cdots + p_na_n^t}{P_n}}{t}\right] \qquad \text{(which is 0/0 for } t = 0\text{)}$$

$$= \lim_{t\to 0}\frac{\dfrac{P_n}{p_1a_1^t + \cdots + p_na_n^t}\left[\dfrac{p_1a_1^t\log a_1 + \cdots + p_na_n^t\log a_n}{P_n}\right]}{1}.$$

Now, whether t approaches 0 from above or below, we have $a_i^t \to 1$ and this gives the limit

$$L = \frac{p_1\log a_1 + \cdots + p_n\log a_n}{P_n} = \log\left[a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}\right]^{1/P_n}.$$

Taking antilogs, then, we define

$$f(0) = \lim_{t\to 0} f(t) = \left[a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}\right]^{1/P_n},$$

which we might observe is just the geometric mean of the set of P_n numbers in which each a_i occurs p_i times.

With $f(0)$ suitably defined, we can proceed normally to establish the desired conclusion by showing that the derivative $f'(t)$ is positive for $t < 0$ and $t > 0$. Again we approach the matter through the logarithm of $f(t)$.

Since the p_i and a_i are all positive numbers, $f(t)$ is also positive for all values of t. Hence the derivative D of $\log f(t)$,

$$D = \frac{[d\log f(t)]}{dt} = \frac{f'(t)}{f(t)},$$

always has the *same sign as* $f'(t)$, unless both are 0. This is not altered by multiplying through by a nonnegative number t^2, and so let us consider the function $F(t)$, defined by

$$F(t) = t^2D = t^2\frac{f'(t)}{f(t)}.$$

A positive value of $F(t)$ can occur only when D is positive, in which case D and $f'(t)$ have the same sign. Thus, *if* we can show that $F(t)$ is positive when t is not zero, it would follow that D, and therefore the desired $f'(t)$, is positive for t not equal to zero. The reason for considering $F(t)$ is that it is easier to work with than $f'(t)/f(t)$.

From $F(t) = t^2 D = t^2 \cdot \frac{d[\log f(t)]}{dt}$, we have

$$F(t) = t^2 \frac{d}{dt}\left[\frac{1}{t}\log\frac{p_1 a_1^t + \cdots + p_n a_n^t}{P_n}\right]$$

$$= t^2\left[\frac{1}{t}\left[\frac{P_n}{p_1 a_1^t + \cdots + p_n a_n^t}\cdot\frac{p_1 a_1^t \log a_1 + \cdots + p_n a_n^t \log a_n}{P_n}\right]\right]$$

$$+ t^2\left(-\frac{1}{t^2}\right)\cdot\log\frac{p_1 a_1^t + \cdots + p_n a_n^t}{P_n}$$

$$= t\cdot\frac{p_1 a_1^t \log a_1 + \cdots + p_n a_n^t \log a_n}{p_1 a_1^t + \cdots p_n a_n^t} - \log\frac{p_1 a_1^t + \cdots p_n a_n^t}{P_n},$$

that is,

$$F(t) = t\cdot\frac{\sum p_i a_i^t \log a_i}{\sum p_i a_i^t} - \log\frac{\sum p_i a_i^t}{P_n}.$$

Recall that all we want to do is show that $F(t)$ is positive for $t \neq 0$. This can be achieved nicely by again appealing to the derivative. As we shall see, it turns out that, for $t \neq 0$, $F'(t)$ has the same sign as t itself, implying that $F(t)$ is a decreasing function for negative t and an increasing function for positive t (see figure). Thus $F(t)$ has a minimum only at $t = 0$, and in the event that $F(0)$ is nonnegative, $F(t)$ would be strictly positive for all $t \neq 0$. But clearly $F(0)$ *is* nonnegative, for setting $t = 0$ in the above expression immediately gives

$$F(0) = 0 - \log 1 = 0.$$

It remains, then, only to calculate the derivative of $F(t)$ and show that it has the same sign as t for $t \neq 0$.

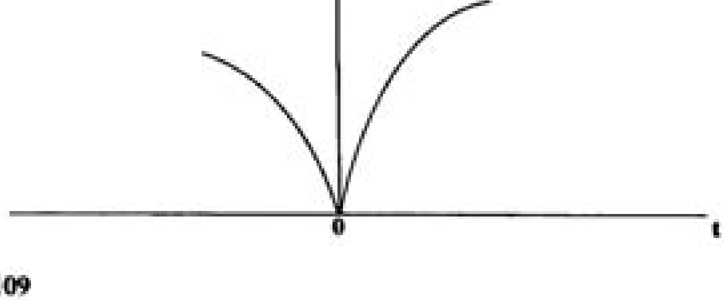


FIGURE 109

Differentiating $F(t)$, we obtain in a straightforward manner that

$$F'(t) = t \cdot \frac{(\sum p_i a_i^t)(\sum p_i a_i^t \log^2 a_i) - (\sum p_i a_i^t \log a_i)(\sum p_i a_i^t \log a_i)}{(\sum p_i a_i^t)^2}$$

$$+ (1) \cdot \frac{\sum p_i a_i^t \log a_i}{\sum p_i a_i^t} - \frac{P_n}{\sum p_i a_i^t} \cdot \frac{\sum p_i a_i^t \log a_i}{P_n}$$

$$= t \cdot \frac{(\sum p_i a_i^t)(\sum p_i a_i^t \log^2 a_i) - (\sum p_i a_i^t \log a_i)^2}{(\sum p_i a_i^t)^2}$$

The denominator here is clearly positive, and as we shall see, it follows from the Cauchy inequality that the numerator is certainly nonnegative. Recall that the Cauchy inequality concerns vectors

$$u = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad v = (v_1, v_2, \ldots, v_n),$$

making the claims that

$$|u \cdot v| = |u|\,|v| \cos\theta, \quad \text{where } u \text{ and } v \text{ are inclined at an angle } \theta$$
$$\leq |u|\,|v|,$$

which, when squared, asserts that

$$(u_1 v_1 + u_2 v_2 + \cdots + u_n v_n)^2 \leq (u_1^2 + u_2^2 + \cdots + u_n^2)(v_1^2 + v_2^2 + \cdots + v_n^2),$$

with equality if and only if $\cos\theta = 1$,
i.e., if and only if the u_i and v_i are proportional for all i.

Now, for

$$u_i = \sqrt{p_i a_i^t} \quad \text{and} \quad v_i = \sqrt{p_i a_i^t \log^2 a_i},$$

we have, for the numerator in the expression for $F'(t)$,

$$(\sum p_i a_i^t)(\sum p_i a_i^t \log^2 a_i) - (\sum p_i a_i^t \log a_i)^2 \geq 0,$$

with equality if and only if u_i and v_i are proportional for all i.
Hence equality occurs if and only if

$$\frac{v_1}{u_1} = \frac{v_2}{u_2} = \cdots = \frac{v_n}{u_n},$$

that is, noting that $v_i = u_i \log a_i$, if and only if

$$\log a_1 = \log a_2 = \cdots \log a_n,$$

which implies

$$a_1 = a_2 = \cdots = a_n.$$

When the a's are *not* all equal, as in the case at hand, then

$$F'(t) = t \cdot (\text{a positive fraction}),$$

and we have the final conclusion that $F'(t)$ and t have the same sign for $t \neq 0$.

In closing, note that the well-known arithmetic mean-geometric mean-harmonic mean inequality is simply a special case of this powerful result. Since $-1 < 0 < 1$, then

$$f(-1) < f(0) < f(1), \text{when the } a\text{'s are not all equal.}$$

But

$$f(1) = \frac{p_1a_1 + p_2a_2 + \cdots + p_na_n}{P_n},$$

the arithmetic mean A of the set of P_n numbers in which a_i occurs p_i times; and, as we have already seen, $f(0)$ is the geometric mean G of these numbers, and finally

$$f(-1) = \frac{P_n}{\dfrac{p_1}{a_1} + \dfrac{p_2}{a_2} \cdots + \dfrac{p_n}{a_n}},$$

the harmonic mean H of these numbers. Hence

$$H \leq G \leq A,$$

with equality if and only if all the a's are equal.

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Subject Index

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Mathematics/Problem Solving

From Erdös to Kiev

Ross Honsberger was born in Toronto, Canada, in 1929 and attended the University of Toronto. After more than a decade of teaching mathematics in Toronto, he took advantage of a sabbatical leave to continue his studies at the University of Waterloo, Canada. He joined its faculty in 1964 in the Department of Combinatorics and Optimization, and has been there ever since. He has published eight best-selling books with the Mathematical Association of America, including his *Mathematical Gems I, II, and III*, *Mathematical Morsels*, *More Mathematical Morsels*, *Mathematical Plums*, *Ingenuity in Mathematics*, and *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*.

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